VIRTUAL STRINGS AND THEIR COBORDISMS

VLADIMIR TURAEV

Abstract. A virtual string is a scheme of self-intersections of a closed curve on a surface. We study algebraic invariants of strings as well as two equivalence relations on the set of strings: homotopy and cobordism. We show that the homotopy invariants of strings form an infinite dimensional Lie group. We also discuss connections between virtual strings and virtual knots.

Contents

1. Introduction

A virtual string is a scheme of self-intersections of a generic oriented closed curve on an oriented surface. More precisely, a virtual string of rank $m \geq 0$ is an oriented circle with $2m$ distinguished points partitioned into m ordered pairs. These m ordered pairs of points are called arrows of the virtual string. An example of a virtual string of rank 3 is shown on Figure 1 where the arrows are represented by geometric vectors.

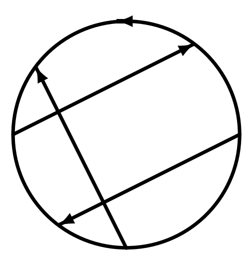

Figure 1. A virtual string of rank 3

A (generic oriented) closed curve on an oriented surface gives rise to an “underlying” virtual string whose arrows correspond to the self-crossings of the curve. The usual homotopy of curves on surfaces suggests a notion of homotopy for strings. The homotopy of curves in 3-manifolds with boundary suggests a notion of cobordism for strings. The main objective of the theory of virtual strings is a study (and eventually classification) of their homotopy classes and cobordism classes. To this end, we introduce algebraic invariants of virtual strings, specifically, a one-variable polynomial u and a so-called based matrix. We formulate obstructions to homotopy/cobordism of strings in terms of these invariants. This leads us to a purely algebraic study of analogues of homotopy and cobordism for skew-symmetric matrices.

As an instance of cobordism, we call a string *slice* if it can be realized by a closed curve on the boundary of an orientable 3-manifold M that is contractible in M. We formulate obstructions to the sliceness of a string in terms of the polynomial u and the based matrix.

We introduce a natural Lie cobracket in the free abelian group generated by the homotopy classes of strings. Dually, the abelian group of $\mathbb{Z}$-valued homotopy invariants of strings becomes a Lie algebra. This Lie algebra is integrated into an infinite dimensional Lie group. This Lie group gives rise to further algebraic objects including a Hopf algebra structure on the (commutative) polynomial algebra generated by the homotopy classes of strings.

Virtual strings are closely related to virtual knots introduced by L. Kauffman [Ka]. In particular, the term "virtual knots" suggested to us the term virtual strings. Virtual knots can be defined as equivalence classes of arrow diagrams which are just virtual strings whose arrows are provided with signs $+$ or $-$. Forgetting these signs, we obtain a map from the set of virtual knots into the set of homotopy classes of virtual strings. We give a more elaborate construction which associates with each virtual knot a polynomial expression in virtual strings with coefficients in the ring $\mathbb{Q}[z]$. This leads to an isomorphism between a "skein algebra" of virtual knots and a polynomial algebra generated by the homotopy classes of strings.

We also study "open strings" which are schemes of self-intersections of generic paths on surfaces.

As an application of this work, we obtain a new interesting relation of cobordism for skew-symmetric matrices and a homeomorphism invariant of knots in cylinders over oriented surfaces with values in the polynomial ring $\mathbb{Q}[z, t]$.

A number of ideas and results of this paper have predecessors in the literature. The homotopy for virtual strings can be defined in terms of the stable equivalence of curves on surfaces introduced by J. S. Carter, S. Kamada, and M. Saito [CKS]. These authors also defined the notions of cobordism and sliceness for curves on surfaces which is essentially equivalent to our cobordism and sliceness for strings. Carter [Ca2] showed that there are closed curves on closed oriented surfaces that bound no singular disks in 3-manifolds bounded by these surfaces. In our language this means that there are non-slice strings (see also [HK]).

The main novelty of the present paper lies in the introduction of new homotopy invariants of strings and new operations on the homotopy classes of strings.

The organization of the paper should be clear from the Contents above.

2. Generalities on virtual strings

2.1. **Definitions.** We give here a formal definition of a virtual string. For an integer $m \geq 0$, a *virtual string* α *of rank* m (or briefly a *string*) is an oriented circle, S, called the *core circle* of α, and a distinguished set of $2m$ distinct points of S partitioned into m ordered pairs. We call these m ordered pairs of points the *arrows* of α. The set of arrows of α is denoted $\mathrm{arr}(\alpha)$. The endpoints $a, b \in S$ of an arrow $(a, b) \in \mathrm{arr}(\alpha)$ are called its *tail* and *head*, respectively. The $2m$ distinguished points of S are called the *endpoints* of α.

The string formed by an oriented circle and an empty set of arrows is called a *trivial virtual string*. An example of a virtual string of rank 3 is shown on Figure 1.

By a *homeomorphism* of two virtual strings, we mean an orientation-preserving homeomorphism of the core circles transforming the set of arrows of the first string onto the set of arrows of the second string. Two virtual strings are *homeomorphic* if they are related by a homeomorphism. Clearly, homeomorphic strings have the same rank.

By abuse of language, the homeomorphism classes of virtual strings will be also called virtual strings.

2.2. **From curves to strings.** By a surface, we mean a smooth *oriented* 2-dimensional manifold. By a *closed curve* on a surface Σ, we mean a generic smooth immersion ω of an oriented circle S into Σ. Recall that a smooth map $S \to \Sigma$ is an *immersion* if its differential is non-zero at all points of S. An immersion $\omega : S \to \Sigma$ is *generic* if $\#(\omega^{-1}(x)) \leq 2$ for all $x \in \Sigma$, the set $\{x \in \Sigma \,|\, \#(\omega^{-1}(x)) = 2\}$ is finite, and all its points are transverse intersections of two branches. Here and below the symbol $\#(A)$ denotes the cardinality of a set A. The points $x \in \Sigma$ such that $\#(\omega^{-1}(x)) = 2$ are called *double points* or *crossings* of ω.

A closed curve $\omega : S \to \Sigma$ gives rise to an *underlying virtual string* α_ω. The core circle of α_ω is S and the arrows of α_ω are all ordered pairs $a, b \in S$ such that $\omega(a) = \omega(b)$ and the pair (a positive tangent vector of ω at a, a positive tangent vector of ω at b) is a positive basis in the tangent space of $\omega(a)$. For instance, the underlying string of a simple closed curve on Σ is a trivial virtual string.

We say that a virtual string is *realized* by a closed curve $\omega : S \to \Sigma$ if it is homeomorphic to α_ω. As we shall see below, every virtual string can be realized by a closed curve on a surface.

2.3. **Homotopy of strings.** The usual homotopy of closed curves on a surface suggests to introduce a relation of homotopy for virtual strings. Observe first that two homotopic curves on a surface can be related by a finite sequence of the following "elementary" moves (and the inverse moves):

(a) a local move adding a small curl to the curve;
(b) a local move pushing a branch of the curve across another branch and creating two new double points;
(c) a local move pushing a branch of the curve across a double point;
(d) ambient isotopy in the surface.

The move (a) has two forms $(a)^+$ and $(a)^-$ depending on whether the curl lies on the left or the right of the curve where the left and the right are determined by the direction of the curve and the orientation of the surface. Considered up to ambient isotopy, the move (b) has three forms depending on the direction of the two branches. Similarly, considered up to ambient isotopy, the move (c) has two forms $(c)^+$ and $(c)^-$ depending on the direction of the branches. Using the standard braid generators σ_1, σ_2 on 3 strands we can encode this move as $\sigma_1\sigma_2\sigma_1 \mapsto \sigma_2\sigma_1\sigma_2$ where the over/undercrossing information is forgotten. The moves $(c)^+$ and $(c)^-$ are obtained by directing (before and after the move) the first and third strands up and the second strand up or down, respectively. It is easy to see that $(c)^+$, $(c)^-$ can be obtained from each other using ambient isotopy, moves (b), and inverses to (b). Similarly, the moves $(a)^+$, $(a)^-$ can be obtained from each other using ambient isotopy, moves (b), $(c)^-$, and inverses to (b). Thus the moves $(a)^-$, (b), $(c)^-$ generate all the other moves.

It is clear that ambient isotopy of a closed curve does not change the underlying virtual string. We now describe the analogues for virtual strings of the moves $(a)^-$, (b), $(c)^-$. In this description and in the sequel, by an *arc* on an oriented circle S we mean an *embedded arc* on S. The orientation of S induces an orientation of all arcs on S. For two distinct points $a, b \in S$, we write ab for the unique oriented arc in S which begins in a and terminates in b. Clearly, $S = ab \cup ba$ and $ab \cap ba = \{a, b\}$.

Let α be a virtual string with core circle S. Pick two distinct points $a, b \in S$ such that the arc $ab \subset S$ is disjoint from the set of endpoints of α. The move $(a)_s$, where s stands for "string", adds to α the pair (a, b). This amounts to attaching a small arrow to S such that the arc in S leading from its tail to its head is disjoint from the endpoints of α. The move $(b)_s$ acts on α as follows. Pick two arcs on S disjoint from each other and from the endpoints of α. Let a, a' be the endpoints of the first arc (in an arbitrary order) and b, b' be the endpoints of the second arc. The move $(b)_s$ adds to α two arrows (a, b) and (b', a'). (This move has four forms depending on the two possible choices for a and two possible choices for b. However, two of these forms of $(b)_s$ are equivalent.) The move $(c)_s$ applies to α when α has three arrows $(a^+, b), (b^+, c), (c^+, a)$ where $a, a^+, b, b^+, c, c^+ \in S$ such that the arcs aa^+, bb^+, cc^+ are disjoint from each other and from the other endpoints of α. The move $(c)_s$ replaces the arrows $(a^+, b), (b^+, c), (c^+, a)$ with the arrows $(a, b^+), (b, c^+), (c, a^+)$.

We say that two virtual strings are *homotopic* if they can be related by a finite sequence of homeomorphisms, the *homotopy moves* $(a)_s$, $(b)_s$, $(c)_s$, and the inverse moves. A virtual string homotopic to a trivial virtual string is said to be *homotopically trivial*. For instance, as it follows directly from the definitions, all virtual strings of rank ≤ 2 are homotopically trivial.

It is clear from what was said above that the underlying virtual strings of homotopic closed curves on a surface are themselves homotopic.

2.4. **Transformations of strings.** For a string α, we define the *opposite string* α^- to be α with opposite orientation on the core circle. The *inverse string* $\overline{\alpha}$ is obtained from α by reversing all its arrows. On the level of closed curves on surfaces, these two transformations correspond to traversing the same curve in the opposite direction and to inverting the orientation of the ambient surface, respectively. If two strings are homotopic, then their opposite (resp. inverse) strings are homotopic.

One can raise a number of questions concerning the transformations $\alpha \mapsto \alpha^-$, $\alpha \mapsto \overline{\alpha}$, $\alpha \mapsto \overline{\alpha}^-$. For instance, one can ask whether there is a string α that is not homotopic to α^- (resp. to $\overline{\alpha}$, $\overline{\alpha}^-$). Below we will answer this question in the positive.

A virtual string α with core circle S is a *product* of virtual strings α_1 and α_2 if there are disjoint arcs $a_1b_1, a_2b_2 \subset S$ such that each arrow of α has both endpoints on either a_1b_1 or on a_2b_2 and the string formed by S and the arrows of α with endpoints on a_ib_i is homeomorphic to α_i for $i = 1, 2$. One can ask whether the product is a well-defined operation on strings (at least up to homotopy) and whether it is commutative. Below we will answer these questions in the negative.

2.5. **Geometric invariants of strings.** We define three geometric characteristics of strings: the genus, the homotopy genus, and the homotopy rank. The *genus* $g(\alpha)$ of a string α is the minimal integer $g \geq 0$ such that α can be realized by a closed curve on a surface of genus g. The *homotopy genus* $hg(\alpha)$ is the minimal integer $g \geq 0$ such that α is homotopic to a string of genus g. The *homotopy rank* $hr(\alpha)$ is the minimal integer $m \geq 0$ such that α is homotopic to a string of rank m. For example, if α is a trivial string, then $g(\alpha) = hg(\alpha) = hr(\alpha) = 0$. It is clear that the homotopy genus and the homotopy rank are homotopy invariants of strings. Below we compute the genus explicitly and show that it is not a homotopy invariant.

The numbers $g(\alpha), hg(\alpha), hr(\alpha)$ are preserved under the transformations $\alpha \mapsto \alpha^-, \alpha \mapsto \overline{\alpha}$.

2.6. **Encoding of strings.** There are two simple methods allowing to encode virtual strings in a compact way. Although we do not use these methods in this paper, we briefly describe them for completeness.

(1) Consider a finite set E consisting of m elements and its disjoint copy $E^+ = \{x^+ \,|\, x \in E\}$. Let $y_1, y_2, \ldots, y_{2m}$ be a sequence of elements of the set $E \cup E^+$ in which every element appears exactly once. (Such a sequence determines a total order in $E \cup E^+$ and vice versa.) The sequence $y_1, y_2, \ldots, y_{2m}$ defines a string of rank m whose underlying circle is $S = \mathbb{R} \cup \{\infty\}$ with right-handed orientation on $\mathbb{R}$ and whose m arrows are the pairs (a, b) such that $a, b \in \{1, 2, \ldots, 2m\} \subset S, y_a \in E$, and $y_b = y_a^+ \in E^+$. Any string can be encoded in this way. For instance, the string drawn in Figure 1 is encoded by the sequence x^+, y, z^+, x, z, y^+ where $E = \{x, y, z\}$.

(2) By Section 2.2, virtual strings can be encoded by closed curves on surfaces. This has an extension similar to Kauffman's graphical encoding of virtual knots in [Ka]. Namely, consider a (generic) closed curve on a surface and suppose that some of its crossings are marked as "virtual". Take the string of this curve as in Section 2.2 and forget all its arrows corresponding to virtual crossings. It is easy to see that every virtual string can be obtained in this way from a closed curve in $\mathbb{R}^2$ with virtual crossings. This yields a graphical encoding of strings by plane curves with virtual crossings. The relation of homotopy for strings has a simple description in this language: it is generated by the moves shown in [Ka], Figure 2 (where the over/undercrossing information should be forgotten).

2.7. **Remarks.** 1. We can point out certain classes of closed curves on surfaces whose underlying virtual strings are homotopically trivial. Since all closed curves on S^2 are contractible, their underlying strings are homotopically trivial. Therefore the same is true for closed curves on any subsurface of S^2, i.e., on any surface of genus 0. In particular, all closed curves on an annulus have homotopically trivial underlying strings. Since each closed curve on a torus can be deformed into an annulus, its underlying string is homotopically trivial. The same holds for closed curves on a torus with holes.

2. The move $(a)_s$ has a version $(a)_s^+$ which is defined as $(a)_s$ above but adds the arrow (b, a) rather than (a, b). This move underlies the move $(a)^+$ on closed curves. The move $(a)_s^+$ preserves the homotopy class of a string. Indeed, it can be expressed as a composition of $(b)_s$, $(c)_s$, and an inverse to $(a)_s$.

3. The move $(c)_s$ has a version $(c)_s^+$ which applies to a string when it has three arrows $(a, b), (a^+, c)$, (b^+, c^+) such that the arcs aa^+, bb^+, cc^+ are disjoint from each other and from the other endpoints of the string. The move $(c)_s^+$ replaces these three arrows with the arrows $(a^+, b^+), (a, c^+), (b, c)$. This move underlies the move $(c)^+$ on closed curves. The move $(c)_s^+$ can be expressed as a composition of $(c)_s^-$ and $(b)_s$.

3. Polynomial u

3.1. **Invariants $\{u_k\}_k$.** Let α be a virtual string with core circle S. Each arrow $e = (a, b) \in \operatorname{arr}(\alpha)$ splits S into two arcs ab and ba. We say that an arrow $f = (c, d)$ of α (distinct from e) *links* e if one of its endpoints lies on ab and the other one lies on ba. More precisely, $f = (c, d)$ links e *positively* (resp. *negatively*) if $c \in ab, d \in ba$ (respectively, if $c \in ba, d \in ab$). If f does not link e, then e and f are *unlinked*. Let $n(e) \in \mathbb{Z}$ be the algebraic number of arrows of α linking e, i.e., the number of arrows of α linking e positively minus the number of arrows of α linking e negatively.

It is easy to trace the behaviour of $n(e)$ under the homotopy moves $(a)_s$, $(b)_s$, $(c)_s$ on α. The move $(a)_s$ adds an arrow e_0 with $n(e_0) = 0$ and keeps $n(e)$ for all other arrows. The move $(b)_s$ adds two arrows e_1, e_2 with $n(e_1) = -n(e_2)$ and keeps $n(e)$ for all other arrows. Consider the move $(c)_s$ and use the notation of Section 2.3. It is obvious that for all arrows e preserved under the move, the number $n(e)$ is also preserved. Each arrow $e = (a^+, b), (b^+, c), (c^+, a)$ occuring before the move gives rise to an arrow $e' = (a, b^+), (b, c^+), (c, a^+)$, respectively, occuring after the move. We claim that $n(e) = n(e')$. Consider for concreteness $e = (a^+, b)$. Note that the points c, c^+ lie either on ab or on ba. Suppose that $c, c^+ \in ab$. Then the arrows (b^+, c) and

(c^+, a) contribute 1 and -1 to $n(e)$, respectively, while the corresponding arrows (b, c^+) and (c, a^+) contribute 0 to $n(e')$. All other arrows contribute the same to $n(e)$ and $n(e')$. Hence $n(e) = n(e')$. If $c, c^+ \in ba$, then the arrows (b^+, c) and (c^+, a) contribute 0 to $n(e)$ while the corresponding arrows (b, c^+) and (c, a^+) contribute -1 and 1 to $n(e')$, respectively. All other arrows contribute the same to $n(e)$ and $n(e')$. Hence $n(e) = n(e')$.

For an integer $k \geq 1$, set

$$u_k(\alpha) = \#\{e \in \operatorname{arr}(\alpha) \,|\, n(e) = k\} - \#\{e \in \operatorname{arr}(\alpha) \,|\, n(e) = -k\} \in \mathbb{Z}.$$

It is clear from what was said above that $u_k(\alpha)$ is preserved under the moves $(a)_s$, $(b)_s$, $(c)_s$. In other words, $u_k(\alpha)$ is a homotopy invariant of α. Clearly, $u_k(\alpha) = 0$ for all k greater than or equal to the rank of α. If α is homotopically trivial, then $u_k(\alpha) = 0$ for all $k \geq 1$.

3.2. **Polynomial $u(\alpha)$.** We can combine the invariants u_k of a virtual string α into a polynomial

$$u(\alpha) = \sum_{k \geq 1} u_k(\alpha)\, t^k$$

where t is a variable. The free term of this polynomial is always 0 and its degree is bounded from above by $m - 1$ where m is the rank of α. This polynomial is a homotopy invariant of α. If α is homotopically trivial, then $u(\alpha) = 0$. (The converse is not true, as we shall see below.) The polynomial $u(\alpha)$ yields an estimate for the homotopy rank $hr(\alpha)$ of α defined in Section 2.5:

$$hr(\alpha) \geq \deg u(\alpha) + 1. \tag{3.2.1}$$

We can rewrite $u(\alpha)$ as follows:

$$u(\alpha) = \sum_{e \in \operatorname{arr}(\alpha), n(e) \neq 0} \operatorname{sign}(n(e))\, t^{|n(e)|} \tag{3.2.2}$$

where $\operatorname{sign}(n) = 1$ for positive $n \in \mathbb{Z}$ and $\operatorname{sign}(n) = -1$ for negative $n \in \mathbb{Z}$. Therefore

$$\sum_{k \geq 1} k\, u_k(\alpha)\, t^{k-1} = u'(\alpha) = \sum_{e \in \operatorname{arr}(\alpha), n(e) \neq 0} n(e)\, t^{|n(e)|-1} = \sum_{e \in \operatorname{arr}(\alpha)} n(e)\, t^{|n(e)|-1}.$$

Substituting $t = 1$, we obtain

$$\sum_{k \geq 1} k\, u_k(\alpha) = u'(1) = \sum_{e \in \operatorname{arr}(\alpha)} n(e) = 0.$$

The last equality follows from the fact that if an arrow f links an arrow e positively, then e links f negatively.

3.3. **Examples.** 1. For positive integers p, q, we define $\alpha_{p,q}$ to be the lattice-looking virtual string formed by a Euclidean circle in $\mathbb{R}^2$ with counterclockwise orientation, p disjoint vertical arrows $e_1, \ldots, e_p$ directed upward and numerated from left to right, and q disjoint horizontal arrows $e_{p+1}, \ldots, e_{p+q}$ crossing $e_1, \ldots, e_p$ from right to left and numerated from bottom to top. (Here we identify arrows with geometric vectors in $\mathbb{R}^2$ connecting two points of the core circle; the numeration of the arrows is compatible with the counterclockwise order of their tails.) Clearly, $n(e_i) = q$ for $i = 1, \ldots, p$ and $n(e_{p+j}) = -p$ for $j = 1, \ldots, q$. Hence $u(\alpha_{p,q}) = pt^q - qt^p$. We conclude that the strings $\{\alpha_{p,q}\}_{p \neq q}$ are pairwise non-homotopic and homotopically non-trivial. The string $\alpha_{1,1}$ is homotopically trivial: it is obtained from a trivial string by $(b)_s$. For $p \geq 2$, we have $u(\alpha_{p,p}) = 0$. However $\alpha_{p,p}$ is homotopically non-trivial as will be shown below.

It follows from the definitions that $\overline{\alpha_{p,q}} = \alpha_{p,q}$ and $(\alpha_{p,q})^- = \alpha_{q,p}$. Thus the string $\alpha = \alpha_{p,q}$ with $p \neq q$ is not homotopic to $\alpha^-, \overline{\alpha}^-$.

Formula 3.2.1 implies that the strings $\alpha_{p,1}$ and $\alpha_{1,p}$ with $p \geq 2$ have minimal rank in their homotopy classes. We shall prove below that the same holds for all $\alpha_{p,q}$ except $\alpha_{1,1}$.

2. A permutation σ of the set $\{1, 2, \ldots, m\}$ gives rise to a virtual string α_σ of rank m as follows. Let $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ be the unit circle with counterclockwise orientation. For $i = 1, \ldots, m$, let a_i (resp. b_i) be the point of S^1 with real part $(i-1)/m$ and negative (resp. positive) imaginary part. Then α_σ is formed by S^1 and the m arrows $\{(a_i, b_{\sigma(i)})\}_{i=1}^m$. For the i-th arrow $e_i = (a_i, b_{\sigma(i)})$,

$$n(e_i) = \sigma(i) - i. \tag{3.3.1}$$

This allows us to compute the polynomial $u(\alpha_\sigma)$ directly from σ. This example generalizes the previous one since $\alpha_{p,q} = \alpha_\sigma$ for the permutation σ of the set $\{1, 2, \ldots, p+q\}$ given by

$$\sigma(i) = \begin{cases} i+q, & \text{if } 1 \leq i \leq p \\ i-p, & \text{if } p < i \leq p+q. \end{cases}$$

3.4. **Properties of u.** We point out a few simple properties of the polynomial u. For a virtual string α, we have $u(\alpha) = u(\overline{\alpha})$. This follows from the fact that if two arrows are linked positively (resp. negatively), then the reversed arrows are also linked positively (resp. negatively). The transformation $\alpha \mapsto \alpha^-$ transforms positively linked pairs of arrows into negatively linked pairs and vice versa. Therefore $u(\alpha^-) = -u(\alpha)$. As an application, we observe that if $u(\alpha) \neq 0$, then α is not homotopic to $\alpha^-, \overline{\alpha}^-$.

It is obvious that if a string α is a product of strings α_1 and α_2, then $u(\alpha) = u(\alpha_1) + u(\alpha_2)$.

Theorem 3.4.1. *An integral polynomial $u(t)$ can be realized as the u-polynomial of a virtual string if and only if $u(0) = u'(1) = 0$.*

Proof. We need only to prove the sufficiency of the condition $u(0) = u'(1) = 0$. The proof goes by induction on the degree of u. If this degree is ≤ 1, then $u = 0$ is realized by a trivial virtual string. Assume that our claim is true for polynomials of degree $< m$ where $m \geq 2$. Let $u(t)$ be a polynomial of degree m with highest term at^m where $a \in \mathbb{Z}$ and $a \neq 0$. Then $v(t) = u(t) - a(t^m - mt)$ is a polynomial of degree $< m$ with $v(0) = v'(1) = 0$. By the inductive assumption, $v(t)$ is realizable as the u-polynomial of a string. By Example 3.3, the polynomial $t^m - mt$ is also realizable. Taking a product of strings we observe that the sum of realizable polynomials is realizable. Hence for $a > 0$, the polynomial $u(t) = v(t) + a(t^m - mt)$ is realizable. If $a < 0$, then this argument shows that $-u(t)$ is realizable by a string, α. Then $u(t)$ is realized by α^-. □

3.5. **Computation for curves.** We compute the polynomial u for the string $\alpha = \alpha_\omega$ underlying a closed curve $\omega : S \to \Sigma$ on a surface Σ. The computation goes in terms of the homological intersection form $B : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ determined by the orientation of Σ. Here and below $H_1(\Sigma) = H_1(\Sigma; \mathbb{Z})$.

Let $e = (a, b)$ be an arrow of α. Then $\omega(a) = \omega(b)$ so that ω transforms the arcs $ab, ba \subset S$ into loops $\omega(ab), \omega(ba)$ in Σ. Set $[e] = [\omega(ab)] \in H_1(\Sigma)$ and $[e]^* = [\omega(ba)] \in H_1(\Sigma)$ where the square brackets on the right-hand side stand for the homology class of a loop. We compute the intersection number $B([e], [e]^*) \in \mathbb{Z}$. The loops $\omega(ab), \omega(ba)$ intersect transversely except at their common origin $\omega(a) = \omega(b)$. Drawing a picture of $\omega(ab), \omega(ba)$ in a neighborhood of $\omega(a) = \omega(b)$, one observes that a small deformation makes these loops disjoint in this neighborhood. The transversal intersections of $\omega(ab), \omega(ba)$ bijectively correspond to the arrows of α linked with e, i.e., the arrows connecting an interior point of ab with an interior point of ba. The intersection sign at such an intersection is $+1$ if the tail of the corresponding arrow lies on ab and is -1 otherwise. Adding these signs, we obtain that $B([e], [e]^*) = n(e)$. This formula can be rewritten in a more convenient form. Set $s = [\omega] = [\omega(S)] \in H_1(\Sigma)$. Observe that $s = [e] + [e]^*$ and therefore

$$B([e], [e]^*) = B([e], s - [e]) = B([e], s) - B([e], [e]) = B([e], s).$$

Thus

$$n(e) = B([e], s). \tag{3.5.1}$$

Therefore for any $k \geq 1$,

$$u_k(\alpha) = \#\{e \in \operatorname{arr}(\alpha) \mid B([e], s) = k\} - \#\{e \in \operatorname{arr}(\alpha) \mid B([e], s) = -k\}$$

and

$$u(\alpha) = \sum_{e \in \operatorname{arr}(\alpha), B([e],s) \neq 0} \operatorname{sign}(B([e], s))\, t^{|B([e],s)|}.$$

Using the bijective correspondence between the set $\operatorname{arr}(\alpha)$ and the set $\bowtie(\omega)$ of double points of ω, we obtain

$$u(\alpha) = \sum_{x \in \bowtie(\omega), B([\omega_x],s) \neq 0} \operatorname{sign}(B([\omega_x], s))\, t^{|B([\omega_x],s)|} \tag{3.5.2}$$

where for $x \in \bowtie(\omega)$, we let $\omega_x : [0, 1] \to \Sigma$ be the loop beginning at x and following along ω until the first return to x and such that the pair (a positive tangent vector of ω_x at 0, a positive tangent vector of ω_x at 1) is a positive basis in the tangent space of x.

3.6. **Coverings and higher polynomials.** The polynomial u gives rise to a family of polynomial invariants of strings numerated by sequences of positive integers. Their construction is based on the notion of a covering for strings. Let α be a string with core circle S and $r \geq 1$ be an integer. Let $\alpha^{(r)}$ be the string formed by S and the arrows $e \in \operatorname{arr}(\alpha)$ such that $n(e) \in r\mathbb{Z}$. We call $\alpha^{(r)}$ the *r-th covering of* α. If α underlies a closed curve $\omega : S \to \Sigma$ on a surface Σ, then $\alpha^{(r)}$ underlies a lift of α to the r-fold covering of Σ induced by the cohomology class in $H^1(\Sigma; \mathbb{Z}/r\mathbb{Z})$ dual to $[\omega] \in H_1(\Sigma)$. Note that $\alpha^{(1)} = \alpha$.

Lemma 3.6.1. *If strings α and β are homotopic, then $\alpha^{(r)}$ is homotopic to $\beta^{(r)}$ for all $r \geq 1$.*

Proof. If α is obtained from β by the homotopy move $(\mathrm{a})_s$, then the additional arrow e verifies $n(e) = 0$ so that $\alpha^{(r)}$ is obtained from $\beta^{(r)}$ by the move $(\mathrm{a})_s$. If α is obtained from β by the move $(\mathrm{b})_s$, then the additional arrows e_1, e_2 verify $n(e_1) = -n(e_2)$. If $n(e_1) \in r\mathbb{Z}$, then $\alpha^{(r)}$ is obtained from $\beta^{(r)}$ by $(\mathrm{b})_s$; otherwise $\alpha^{(r)} = \beta^{(r)}$. Let α be obtained from β by the move $(\mathrm{c})_s$ replacing three arrows e_1, e_2, e_3 by e_1', e_2', e_3'. It was shown above that $n(e_i') = n(e_i)$ for $i = 1, 2, 3$. It is easy to check that $n(e_1) + n(e_2) + n(e_3) = 0$. Three cases may occur: the numbers $n(e_1), n(e_2), n(e_3)$ are divisible by r; one of these numbers is divisible by r and the other two are not; neither of these numbers is divisible by r. In the first case $\alpha^{(r)}$ is obtained from $\beta^{(r)}$ by $(\mathrm{c})_s$. In the second and third cases $\alpha^{(r)} = \beta^{(r)}$. This implies the claim of the lemma. □

Iterating the coverings, we can define for a string α and a finite sequence of positive integers $r_1, ..., r_k$, a string

$$\alpha^{(r_1,\ldots,r_k)} = (\ldots(\alpha^{(r_1)})^{(r_2)}\ldots)^{(r_k)}.$$

Set

$$u^{r_1,\ldots,r_k}(\alpha) = u(\alpha^{(r_1,\ldots,r_k)}) \in \mathbb{Z}[t].$$

The results above imply that this polynomial is a homotopy invariant of α.

3.7. **Exercises.** 1. Verify that all virtual strings of rank 3 are either homotopically trivial or homeomorphic to $\alpha_{1,2}, \alpha_{2,1}$.

2. For an integer $r \geq 1$ and a virtual string α, define a virtual string $r \cdot \alpha$ as follows. Identifying the core circle of α with $S^1 \subset \mathbb{C}$ we can present arrows of α by geometric vectors with endpoints on S^1. Replace each of these vectors, say e, by r disjoint parallel vectors $e_1, \ldots, e_r$ running closely to e and having endpoints on S^1. This gives a virtual string $r \cdot \alpha$ of rank rm where m is the rank of α. Check that $u(r \cdot \alpha)(t) = r\, u(\alpha)(t^r)$. In particular, if $u(\alpha) \neq 0$, then $r \cdot \alpha$ is homotopically non-trivial.

3. Show that the rank 4 string α_σ with $\sigma = (1342)$ is homotopically trivial. Hint: apply to α_σ the move $((\mathrm{c})_s^+)^{-1}$ as in Remark 2.7.3 where $a = a_2, a^+ = a_3, b = a_4, b^+ = b_4, c = b_2, c^+ = b_1 \in S^1$.

4. Let σ be a permutation of $\{1, 2, \ldots, m\}$. If $\sigma(m) = m$ and τ is the restriction of σ to $\{1, 2, \ldots, m-1\}$, then α_σ is homotopic to α_τ. If $\sigma(1) = m-1, \sigma(2) = 1, \sigma(3) = m, \sigma(m) = 2$, then α_σ is homotopic to α_τ where τ is the permutation of $\{1, 2, \ldots, m-2\}$ defined by $\tau(1) = m-2$ and $\tau(i) = \sigma(i+1) - 1$ for $i > 1$.

5. Let a string α be a product of the strings $\alpha_{1,3}$, $\alpha_{1,4}$, $\alpha_{2,1}$, $\alpha_{2,4}$, $\alpha_{3,5}$, $\alpha_{4,3}$, $\alpha_{5,1}$, $\alpha_{5,2}$. Show that $u(\alpha) = 0$ and $\alpha^{(2)}$ is homotopic to $\alpha_{2,4}$. Thus $u^{(2)}(\alpha) = u(\alpha^{(2)}) = 2t^4 - 4t^2$.

6. More generally, for any $p, q, s, m \geq 1$, if a string α is a product of the strings $\alpha_{s+m,p}$, $\alpha_{p,s}$, $\alpha_{p,m}$, $\alpha_{s+m,q}$, $\alpha_{q,s}$, $\alpha_{q,m}$, $\alpha_{p+q,s+m}$, $\alpha_{s,p+q}$, $\alpha_{m,p+q}$, then $u(\alpha) = 0$. If $p, s \in r\mathbb{Z}$ and q, m are prime to r, then $\alpha^{(r)} = \alpha_{p,s}$.

4. Geometric realization of virtual strings

4.1. **Realization of strings.** We explain here that every virtual string admits a canonical realization by a closed curve on a surface and moreover describe all its realizations.

Let α be a virtual string of rank m with core circle S. Identifying the tail with the head for all arrows of α, we transform S into a 1-dimensional CW-complex $\Gamma = \Gamma_\alpha$. We thicken Γ to a surface Σ_α as follows. If $m = 0$, then $\Gamma = S$ and we set $\Sigma_\alpha = S \times [-1, 1]$. Assume that $m \geq 1$. The 0-cells (vertices) of Γ have valency 4 and their number is equal to m. A neighborhood of a vertex $v \in \Gamma$ embeds into a copy D_v of the unit 2-disk $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$ as follows. Suppose that v is obtained from an arrow (a, b) where $a, b \in S$. Note that any point $x \in S$ splits its small neighborhood in S into two oriented arcs, one of them being incoming and the other one being outgoing with respect to x. A neighborhood of v in Γ consists of four arcs which can be identified with small incoming and outgoing arcs of a, b on S. We embed this neighborhood into D_v so that v goes to the origin and the incoming (resp. outgoing) arcs of a, b go to the intervals $[-1, 0] \times 0$, $0 \times [-1, 0]$ (resp. $[0, 1] \times 0$, $0 \times [0, 1]$), respectively. In this way the vertices of Γ can be thickened to (disjoint) copies of the unit 2-disk endowed with counterclockwise orientation. Each 1-cell of Γ

connects two (possibly coinciding) vertices and can be thickened to a ribbon connecting the corresponding 2-disks. The thickening is uniquely determined by the condition that the orientation of these 2-disks extends to their union with the ribbon. Thickening in this way all the vertices and 1-cells of Γ we embed Γ into a surface Σ_α. By construction, Σ_α is a compact connected oriented surface with non-void boundary and Euler characteristic $\chi(\Sigma_\alpha)=\chi(\Gamma)=-m$. Composing the natural projection $S\to\Gamma$ with the inclusion $\Gamma\hookrightarrow\Sigma_\alpha$ we obtain a closed curve $\omega_\alpha : S\to\Sigma_\alpha$ realizing α. The construction of Σ_α is well known, see [Fr], [Ca1], [CW].

It is clear that for any surface Σ and any (generic) closed curve $\omega : S\to\Sigma$ realizing α, a regular neighborhood of $\omega(S)$ in Σ is homeomorphic to Σ_α. Moreover, the homeomorphism can be chosen to transform ω into ω_α. In other words, ω can be obtained as a composition of ω_α with an orientation-preserving embedding $\Sigma_\alpha\hookrightarrow\Sigma$. In particular, Σ_α is a surface of minimal genus containing a closed curve realizing α. Therefore the genus $g(\alpha)$ of α defined in Section 2.5 is equal to the genus of Σ_α. It will be explicitly computed in the next subsection. Note finally that a closed surface of minimal genus containing a curve realizing α is obtained from Σ_α by gluing 2-disks to all components of $\partial\Sigma_\alpha$.

4.2. **Homological computations.** Consider again a virtual string α of rank m with core circle S. Let $\Gamma=\Gamma_\alpha$, $\Sigma=\Sigma_\alpha$, and $\omega=\omega_\alpha : S\to\Sigma_\alpha$ be the graph, the surface, and the closed curve constructed in the previous subsection. The orientation of Σ determines a homological intersection pairing $B=B_\alpha : H_1(\Sigma)\times H_1(\Sigma)\to\mathbb{Z}$. This bilinear pairing is skew-symmetric and its rank is equal to twice the genus of Σ. Thus

$$g(\alpha)=(1/2)\,\mathrm{rank}B_\alpha.$$

In particular, α can be realized by a closed curve on S^2 or $\mathbb{R}^2$ if and only if $B_\alpha=0$.

Since Γ is a deformation retract of Σ, the inclusion homomorphism $H_1(\Gamma)\to H_1(\Sigma)$ is an isomorphism. Since Γ is a connected graph with $\chi(\Gamma)=-m$, the group $H_1(\Gamma)=H_1(\Sigma)$ is a free abelian group of rank $m+1$. We describe a canonical basis in $H_1(\Sigma)$. Set $s=[\omega]\in H_1(\Sigma)$. For an arrow $e=(a,b)\in\mathrm{arr}(\alpha)$, the map ω transforms the arc $ab\subset S$, leading from a to b in the positive direction, into a loop $\omega(ab)$ in Σ. Set $[e]=[\omega(ab)]\in H_1(\Sigma)$. An easy induction on m shows that $s\cup\{[e]\}_{e\in\mathrm{arr}(\alpha)}$ is a basis of $H_1(\Sigma)$. Our next aim is to compute the matrix of B in this basis. Note for the record that $B(x,y)=-B(y,x)$ and $B(x,x)=0$ for all elements x,y of this basis.

By Formula 3.5.1, $B([e],s)=n(e)$ for any $e\in\mathrm{arr}(\alpha)$. To compute the other values of B, we need more notation. Let a,b be distinct point of S. The *interior* of the arc $ab\subset S$ is the set $(ab)^\circ=ab-\{a,b\}$. For any arcs $ab,cd\subset S$, we define $ab\cdot cd\in\mathbb{Z}$ to be the number of arrows of α with tail in $(ab)^\circ$ and head in $(cd)^\circ$ minus the number of arrows of α with tail in $(cd)^\circ$ and head in $(ab)^\circ$. Note that the arrows with both endpoints in $(ab)^\circ\cap(cd)^\circ$ appear in this expression twice with opposite signs and therefore cancel out. Clearly, $ab\cdot cd=-cd\cdot ab$. In particular, $ab\cdot ab=0$. If $e=(a,b)$ is an arrow of α, then it follows from the definitions that $n(e)=ab\cdot ba$.

Lemma 4.2.1. *Let $e=(a,b)$ and $f=(c,d)$ be two arrows of α. Then $B([e],[f])=ab\cdot cd+\varepsilon$ where $\varepsilon=0$ if e and f are unlinked, $\varepsilon=1$ if f links e positively, and $\varepsilon=-1$ if f links e negatively.*

Proof. If $e=f$, then $a=c,b=d$ and all terms of the stated equality are equal to 0. (Note that an arrow is unlinked with itself.) Assume from now on that $e\neq f$ so that a,b,c,d are pairwise distinct points of S.

Suppose first that e and f are unlinked. There are four cases to consider depending on whether the endpoints of e,f lie on S in the cyclic order (i) a,b,c,d, or (ii) a,b,d,c, or (iii) a,c,d,b, or (iv) a,d,c,b.

In the case (i), the arcs $ab,cd\subset S$ are disjoint so that $[e],[f]\in H_1(\Sigma)$ are represented by transversal loops $\omega(ab),\omega(cd)$, respectively. Then $B([e],[f])=ab\cdot cd$, cf. Section 3.5.

In the case (ii), the arcs $ab,dc\subset S$ are disjoint so that $[e],[f]^*=s-[f]\in H_1(\Sigma)$ are represented by transversal loops $\omega(ab),\omega(dc)$, respectively. Hence $B([e],[f]^*)=ab\cdot dc$ and

$$B([e],[f])=B([e],s-[f]^*)=B([e],s)-B([e],[f]^*)=n(e)-ab\cdot dc=ab\cdot cd.$$

In the case (iii), we have $B([e],[f])=-B([f],[e])=-cd\cdot ab=ab\cdot cd$ since the pair (f,e) satisfies the conditions of (ii).

In the case (iv), the arcs $ba,dc\subset S$ are disjoint so that $[e]^*=s-[e],[f]^*=s-[f]\in H_1(\Sigma)$ are represented by transversal loops $\omega(ba),\omega(dc)$, respectively. Therefore

$$B([e],[f])=B([e],s)+B(s,[f])+B(s-[e],s-[f])=n(e)-n(f)+B([e]^*,[f]^*)=n(e)-n(f)+ba\cdot dc.$$

It remains to observe that

$$ba\cdot dc=-n(e)-ba\cdot cd=-n(e)+cd\cdot ba=-n(e)+n(f)-cd\cdot ab=-n(e)+n(f)+ab\cdot cd.$$

Suppose that f links e positively. Then their endpoints lie on S in the cyclic order a, c, b, d. The loops $X = \omega(ab), Y = \omega(cd)$ representing $[e], [f] \in H_1(\Sigma)$ are not transversal since both contain $\omega(cb)$. Pushing Y slightly to its left in Σ, we obtain a loop, Y^+, transversal to X. It is understood that the point $\omega(c) = \omega(d) \in Y$ is pushed to a point lying between $\omega(ac)$ and $\omega(da)$ in a small neighborhood of $\omega(c) = \omega(d)$. Introducing coordinates (x, y) in this neighborhood we can locally identify X, Y, Y^+ with the axis $y = 0$, the union of two half-lines $x = 0, y \leq 0$ and $y = 0, x \geq 0$, and the union of two half-lines $x = -1, y \leq 1$ and $y = 1, x \geq -1$, respectively. To compute the intersection number $B([e], [f]) = X \cdot Y = X \cdot Y^+$, we split the set $X \cap Y^+$ into four disjoint subsets. The first of them consists of a single point near $\omega(c) = \omega(d)$, given in the coordinates above by $x = -1, y = 0$. This point contributes 1 to $X \cdot Y^+$. The second subset of $X \cap Y^+$ is $\omega(ac) \cap Y^+$; its points are numerated by arrows of α with one endpoint in the interior of ac and the other endpoint in the interior of cd. The contribution of these crossings to $X \cdot Y^+$ is equal to $ac \cdot cd$. The third subset of $X \cap Y^+$ is numerated by the crossings of $\omega(cb)$ with the part of Y^+ obtained by pushing $\omega(bd) \subset Y$ to the left; they are numerated by arrows of α with one endpoint in the interior of cb and the other endpoint in the interior of bd. The contribution of these crossings to $X \cdot Y^+$ is $cb \cdot bd$. The forth subset of $X \cap Y^+$ is numerated by the self-crossings of $\omega(cb)$: each of them gives rise to two points of $X \cap Y^+$ with opposite intersection signs. Therefore this forth subset contributes 0 to $X \cdot Y^+$. Summing up these contributions we obtain

$$B([e], [f]) = 1 + ac \cdot cd + cb \cdot bd + 0 = ac \cdot cd + cb \cdot cb + cb \cdot bd + 1$$

$$= ac \cdot cd + cb \cdot cd + 1 = ab \cdot cd + 1.$$

If f links e negatively, then e links f positively and by the results above,

$$B([e], [f]) = -B([f], [e]) = -(cd \cdot ab + 1) = ab \cdot cd - 1.$$

□

4.3. **Examples.** (1) Consider the string $\alpha = \alpha_{p,q}$ with $p, q \geq 1$ introduced in Section 3.3.1. Recall the arrows $e_1, \ldots, e_{p+q}$ of α. We compute the matrix of the bilinear form $B = B_\alpha : H_1(\Sigma_\alpha) \times H_1(\Sigma_\alpha) \to \mathbb{Z}$ with respect to the basis $s \cup \{[e_i]\}_{i=1}^{p+q}$. By Formula 3.5.1, $B([e_i], s) = q$ for $i = 1, \ldots, p$ and $B([e_{p+j}], s) = -p$ for $j = 1, \ldots, q$. Each pair of arrows $e_i, e_{i'}$ with $i, i' = 1, \ldots, p$ is unlinked and by Lemma 4.2.1, $B([e_i], [e_{i'}]) = 0$. Similarly, each pair of arrows $e_{p+j}, e_{p+j'}$ with $j, j' = 1, \ldots, q$ is unlinked and $B([e_{p+j}], [e_{p+j'}]) = 0$. The arrow e_{p+j} links e_i positively and by Lemma 4.2.1, $B([e_i], [e_{p+j}]) = (p-i) + (q-j) + 1$. It is easy to compute that the rank of B is equal to 2 if $p = q = 1$, to 6 if $\min(p, q) \geq 3$, and to 4 in all the other cases. The genus $g(\alpha)$, as we know, is half of this rank. In particular, $g(\alpha_{1,1}) = 1$ which shows that the genus is not a homotopy invariant.

(2) Consider the string $\alpha = \alpha_\sigma$ defined in Section 3.3.2 for a permutation σ of the set $\{1, 2, \ldots, m\}$. Recall the arrows $e_1, \ldots, e_m$ of α. We compute the matrix of the bilinear form $B = B_\alpha : H_1(\Sigma_\alpha) \times H_1(\Sigma_\alpha) \to \mathbb{Z}$ with respect to the basis $s \cup \{[e_i]\}_{i=1}^m$. The number $B([e_i], s) = n(e_i)$ is computed by Formula 3.3.1. Pick two indices i, j with $1 \leq i < j \leq m$. Lemma 4.2.1 implies that if $\sigma(i) < \sigma(j)$, then

$$B([e_i], [e_j]) = \#\{k \,|\, i < k < j, \sigma(j) < \sigma(k)\} \;-\; \#\{k \,|\, j < k \leq m, \sigma(i) < \sigma(k) < \sigma(j)\}.$$

If $\sigma(j) < \sigma(i)$, then

$$B([e_i], [e_j]) = \#\{k \,|\, i < k < j, \sigma(j) < \sigma(k)\} \;+\; \#\{k \,|\, j < k \leq m, \sigma(j) < \sigma(k) < \sigma(i)\} + 1.$$

5. Cobordism of strings and the slice genus

5.1. **Cobordism of strings.** Two strings α and β are *cobordant* if there are an oriented 3-manifold M and two disjoint closed curves on ∂M realizing α and $\overline{\beta}$, respectively, and homotopic to each other in M. Here two curves on a surface are *disjoint* if their images are disjoint subsets of the surface. The involution $\beta \mapsto \overline{\beta}$ is needed in the definition of cobordism to ensure the reflexivity, cf. the proof of the next lemma.

In the definition of cobordism, we do not require M to be compact or connected. However, replacing M by a regular neighborhood of a homotopy relating the two curves in M we can always assume that M is compact and connected.

Lemma 5.1.1. *Cobordism is an equivalence relation on the set of (homeomorphism classes of) strings. If strings α and β are cobordant, then $\overline{\alpha}$ is cobordant to $\overline{\beta}$ and α^- is cobordant to β^-.*

Proof. To see that a string α is cobordant to itself, we realize α by a closed curve ω on a closed surface Σ and take $M = \Sigma \times [0,1]$ with the orientation obtained as the product of the orientation in Σ and the right-handed orientation in $[0,1]$. Then $\omega \times 1$ and $\omega \times 0$ are disjoint closed curves on $\partial M = (\Sigma \times 1) \cup (-\Sigma \times 0)$ realizing respectively α and $\overline{\alpha}$. These curves are homotopic in M.

The relation of cobordism is symmetric: if two disjoint closed curves on ∂M realize strings $\alpha, \overline{\beta}$, then the same curves on $-\partial M = \partial(-M)$ realize $\overline{\alpha}, \beta$.

Suppose that a string α_1 is cobordant to α_2 and α_2 is cobordant to α_3. Let M, M' be disjoint oriented 3-manifolds such that $\alpha_1, \overline{\alpha}_2$ are realized by disjoint closed curves ω_1, ω_2 on ∂M homotopic in M and $\alpha_2, \overline{\alpha}_3$ are realized by disjoint closed curves ω'_2, ω_3 on $\partial M'$ homotopic in M'. Observe that a regular neighborhood, U, of ω_2 in ∂M is homeomorphic to a regular neighborhood, U', of ω'_2 in $\partial M'$ via an orientation reversing homeomorphism transforming ω_2 into ω'_2. Gluing M to M' along $U \approx U'$ we obtain an oriented 3-manifold N. The curves ω_1, ω_3 lie on ∂N and realize $\alpha_1, \overline{\alpha}_3$, respectively. These curves are disjoint and homotopic in N. Hence α_1 is cobordant to α_3.

The last claim of the lemma is obtained by inverting orientation on the ambient 3-manifold (resp. on the circle). $\square$

Theorem 5.1.2. *Homotopic strings are cobordant.*

Proof. For strings α, β, we write $\alpha \sim \beta$ if these strings can be realized by homotopic (possibly intersecting) closed curves on the same surface. The relation $\sim$ is reflexive and symmetric but not transitive. The following lemma shows that the relation of homotopy is precisely the equivalence relation generated by $\sim$.

Lemma 5.1.3. *Two strings α, β are homotopic if and only if there is a sequence of strings $\alpha_1 = \alpha, \alpha_2, ..., \alpha_n = \beta$ such that $\alpha_i \sim \alpha_{i+1}$ for $i = 1, ..., n-1$.*

Proof. As we know, the underlying virtual strings of homotopic closed curves on a surface are themselves homotopic. Therefore if there is a sequence of strings $\alpha_1 = \alpha, \alpha_2, ..., \alpha_n = \beta$ such that $\alpha_i \sim \alpha_{i+1}$ for $i = 1, ..., n-1$, then α is homotopic to β. To prove the converse, it suffices to show that if β is obtained from α by a homotopy move $(a)_s$, $(b)_s$, or $(c)_s$, then $\alpha \sim \beta$.

Let S be the core circle of α and $\omega : S \to \Sigma$ be a curve realizing α on a surface Σ. Pick distinct points $a, b \in S$ such that the arc $ab \subset S$ does not contain endpoints of α. Let β be obtained from α by the move $(a)_s$ adding the arrow (a, b). Attaching to ω a small curl on the right of $\omega(ab)$, we obtain a closed curve $\omega' : S \to \Sigma$ realizing β. Clearly, ω' is homotopic to ω in Σ. Hence $\alpha \sim \beta$.

Pick two arcs x, y on S disjoint from each other and from the endpoints of α. Let a, a' be the endpoints of x (in an arbitrary order) and b, b' be the endpoints of y. Let β be obtained from α by the move $(b)_s$ adding to α the arrows (a, b) and (b', a'). Let $D_x, D_y \subset \Sigma - \omega(S)$ be two small closed disks lying near the arcs $\omega(x), \omega(y)$, respectively. Removing the interiors of these disks from Σ and gluing the circles $\partial D_x, \partial D_y$ along an orientation reversing homeomorphism, we obtain a new (oriented) surface, Σ', containing $\omega(S)$. In Σ' the arcs $\omega(x)$ and $\omega(y)$ are adjacent to the component of $\Sigma' - \omega(S)$ containing $\partial D_x = \partial D_y$. We can push $\omega(x)$ across this component towards $\omega(y)$ and eventually across $\omega(y)$. This gives a curve $\omega' : S \to \Sigma'$ realizing β and homotopic to $\omega : S \to \Sigma'$. Hence $\alpha \sim \beta$. Note that the four possible forms of the move $(b)_s$ (depending on whether x leads from a to a' or from a' to a and similarly for y) are realized by choosing D_x, D_y on the left or on the right of $\omega(x), \omega(y)$.

Suppose that α has three arrows $(a^+, b), (b^+, c), (c^+, a)$ where $a, a^+, b, b^+, c, c^+ \in S$ such that the (positively oriented) arcs aa^+, bb^+, cc^+ are disjoint from each other and from the other endpoints of α. Let β be obtained from α by the move $(c)_s$ replacing the arrows $(a^+, b), (b^+, c), (c^+, a)$ with the arrows $(a, b^+), (b, c^+), (c, a^+)$. Consider the canonical realization $\omega_\alpha : S \to \Sigma_\alpha$ of α. Observe that the arcs $\omega_\alpha(aa^+), \omega_\alpha(bb^+), \omega_\alpha(cc^+)$ form a simple closed curve in Σ_α isotopic to a component of $\partial\Sigma_\alpha$. Gluing a 2-disk D to this component of $\partial\Sigma_\alpha$ we embed Σ_α into a bigger surface, Σ. Pushing $\omega_\alpha(aa^+)$ across $D \subset \Sigma$ and then across the double point $\omega(b^+) = \omega(c)$, we obtain a curve $\omega' : S \to \Sigma$ realizing β and homotopic to $\omega_\alpha : S \to \Sigma$. Hence $\alpha \sim \beta$. $\square$

We now accomplish the proof of Theorem 5.1.2. The obvious cylinder construction shows that the underlying virtual strings of homotopic curves on a surface are cobordant. Thus if we have strings α, β with $\alpha \sim \beta$, then α is cobordant to β. Lemmas 5.1.1 and 5.1.3 now imply the claim of the theorem. $\square$

Theorem 5.1.4. *The polynomial u is a cobordism invariant of strings.*

Proof. We begin with a lemma.

Lemma 5.1.5. *Let F be a compact (oriented) surface whose boundary is a union of $r \geq 1$ circles $S_1, ..., S_r$ with the induced orientation. Let M be an oriented 3-manifold and $\omega : F \to M$ be a map such that $\omega(\partial F) \subset \partial M$, $\omega(S_i) \cap \omega(S_j) = \emptyset$ for all $i \neq j$, and the restriction of ω to each circle S_i is a generic (closed) curve in ∂M for $i = 1, ..., r$. Let α_i be the virtual string underlying the latter curve. If the genus of F is 0, then*

$$\sum_{i=1}^{r} u(\alpha_i) = 0. \tag{5.1.1}$$

Proof. We need a few general facts about maps $F \to M$. A point $a \in \operatorname{Int} F$ is a *simple branch point* of a map $\omega : F \to M$ if there is a closed 3-ball $D^3 \subset \operatorname{Int} M$ such that $\omega(F) \cap D^3$ is the cone over a figure eight curve in $S^2 = \partial D^3$ with cone point $\omega(a) \in \operatorname{Int} D^3$. Here by a figure eight curve in S^2 we mean a closed curve with one transversal self-intersection. The set of simple branch points of ω is denoted $Br(\omega)$.

Set $\mathbb{R}_+ = \{r \in \mathbb{R}, r \geq 0\}$. We say that a map $\omega : F \to M$ is *generic* if $\omega^{-1}(\partial M) = \partial F$ and

(i) the restriction of ω to ∂F is an immersion into ∂M; any point of $\omega(\partial F)$ has a neighborhood $V \subset M$ such that the pair $(V, V \cap \omega(F))$ is homeomorphic to either $(\mathbb{R}^2, \mathbb{R} \times 0) \times \mathbb{R}_+$ or $(\mathbb{R}^2, \mathbb{R} \times 0 \cup 0 \times \mathbb{R}) \times \mathbb{R}_+$;

(ii) the restriction of ω to $\operatorname{Int} F - Br(\omega)$ is an immersion into $\operatorname{Int} M - \omega(Br(\omega))$; any point of $\omega(\operatorname{Int} F - Br(\omega))$ has a neighborhood $V \subset \operatorname{Int} M$ such that the pair $(V, V \cap \omega(F))$ is homeomorphic to either $(\mathbb{R}^3, \mathbb{R}^2 \times 0)$, or $(\mathbb{R}^3, \mathbb{R}^2 \times 0 \cup 0 \times \mathbb{R}^2)$ or $(\mathbb{R}^3, \mathbb{R}^2 \times 0 \cup 0 \times \mathbb{R}^2 \cup \mathbb{R} \times 0 \times \mathbb{R})$.

Fix a generic map $\omega : F \to M$. Set $T = Br(\omega) \cup \{a \in F \,|\, \#\omega^{-1}(\omega(a)) \geq 2\} \subset F$. It is clear from (i), (ii) that T consists of a finite number of immersed circles in $\operatorname{Int} F$ and immersed proper intervals in F. These circles and intervals have only double transversal crossings. The set of these crossings $\bowtie(T)$ is finite and consists precisely of the preimages of the triple points of ω. The set $\partial T = T \cap \partial F$ is finite and consists precisely of the preimages of the double points of $\omega|_{\partial F} : \partial F \to \partial M$. The set $Br(\omega) \subset T$ is also finite and disjoint from $\bowtie(T)$ and ∂T.

Let $\tilde{T}$ be an abstract 1-dimensional manifold parametrizing T. The projection $p : \tilde{T} \to T$ is 2-to-1 over $\bowtie(T)$ and 1-to-1 over $T - \bowtie(T)$. We shall identify the points of $T - \bowtie(T)$ with their preimages under p.

For any point $a \in T - (\bowtie(T) \cup Br(\omega))$ there is exactly one other point $b \in T - (\bowtie(T) \cup Br(\omega))$ such that $\omega(a) = \omega(b)$. The correspondence $a \leftrightarrow b$ extends by continuity to an involution τ on $\tilde{T}$. The set of fixed points of τ is $Br(\omega)$. It is clear that $\omega\, p\, \tau = \omega\, p : \tilde{T} \to M$. For $a \in \partial T = \partial \tilde{T}$, the point $\tau(a)$ is the unique point $b \in \partial T - \{a\}$ such that $\omega(a) = \omega(b)$.

We define an involution $\mu : \partial T \to \partial T$. For $a \in \partial T = \partial \tilde{T}$, let $I_a \subset \tilde{T}$ be the interval adjacent to a and let $\mu(a) \in \partial T$ be its endpoint distinct from a. Clearly, $\mu^2 = \mathrm{id}$. We claim that μ commutes with $\tau|_{\partial T}$. Indeed, if $\tau(I_a) = I_a$, then τ exchanges the endpoints of I_a so that $\tau = \mu$ on ∂I_a. (In this case τ must have a unique fixed point on I_a, so that I_a contains a unique branch point of ω.) If $\tau(I_a) \neq I_a$, then $\tau(I_a)$ has the endpoints $\tau(a), \tau(\mu(a))$ so that $\mu(\tau(a)) = \tau(\mu(a))$. Since $\tau|_{\partial T}$ and μ commute, μ induces an involution on $\partial T/\tau = \bowtie(\omega|_{\partial F})$. The latter involution is denoted ν.

Assume from now on that ω maps the components $S_1, ..., S_r$ of ∂F to disjoint subsets of ∂M. Each crossing point $x \in \bowtie(\omega|_{\partial F})$ of $\omega|_{\partial F}$ is then a self-crossing of $\omega(S)$ for a certain component $S = S_i$ of ∂F. Consider the loop ω_x in ∂M beginning at x and following along $\omega(S)$ until the first return to x and such that the pair (a positive tangent vector of ω_x at 0, a positive tangent vector of ω_x at 1) is a positive basis in the tangent space of x in ∂M. Let $[\omega_x] \in H_1(\partial M)$ be the homology class of ω_x and $\mathrm{in} : H_1(\partial M) \to H_1(M)$ be the inclusion homomorphism. We have either $\nu(x) = x$ or $\nu(x) \neq x$. We prove that in the first case $\mathrm{in}([\omega_x]) \in \omega_*(H_1(F)) \subset H_1(M)$ and in the second case

$$\mathrm{in}([\omega_x] + [\omega_{\nu(x)}]) \in \omega_*(H_1(F)) \subset H_1(M).$$

Let $\omega^{-1}(x) = \{a, b\}$ where $a, b \in S \cap \partial T$ and the corresponding arrow is directed from a to b as in Section 2.2. Denote the positive arc $ab \subset S$ by γ_x and observe that $\omega_x = \omega(\gamma_x)$. Suppose that $\nu(x) = x$. Then $\mu(a) \in \{a, b\}$. Inspecting the orientations of the sheets of $\omega(F)$ meeting along $\omega(T)$, we observe that $\mu : \partial T \to \partial T$ transforms arrowtails into arrowheads and vice versa (this was first pointed out in [Ca2]). Therefore $\mu(a) = b$. By the definition of τ, we have $\tau(a) = b$ and $\tau(b) = a$. Since τ preserves the set $\partial I_a = \{a, b\}$, we have $\tau(I_a) = I_a$. Observe that the product of the path $\gamma_x = ab \subset S$ with the immersed interval $p(I_a) \subset F$ oriented from b to a is a loop in F, say ρ. The loop $\omega(\rho)$ in M is a product of $\omega(\gamma_x) = \omega_x$ with the loop $\omega p|_{I_a}$. The latter loop has the form $\delta\delta^{-1}$ where δ is the path in M obtained by restricting $\omega\, p$ to the arc in I_a leading from b to the unique branch point of ω on I_a. This loop is contractible in M. Hence $\mathrm{in}([\omega_x]) = [\omega(\rho)] \in \omega_*(H_1(F))$.

Suppose that $\nu(x) \neq x$. Note that the path $\gamma_{\nu(x)}$ begins at $\mu(b)$ and terminates at $\mu(a)$. Consider the loop $\rho = \gamma_x\, p(I_b)\, \gamma_{\nu(x)}\, (p(I_a))^{-1}$ in F beginning and ending at a. Here the intervals I_b, I_a are oriented from b to $\mu(b)$ and from a to $\mu(a)$, respectively. Then $\omega(\rho)$ is the product of the loop ω_x beginning and ending in x, the path $\omega p(I_b)$ beginning in x and ending in $\nu(x)$, the loop $\omega_{\nu(x)}$ beginning and ending in $\nu(x)$, and the path $(\omega p(I_a))^{-1}$ beginning in $\nu(x)$ and ending in x. The paths $\omega p(I_b)$, $(\omega p(I_a))^{-1}$ are mutually inverse since $I_b = \tau(I_a)$ and $\omega p \tau = \omega p$. Hence

$$\mathrm{in}([\omega_x] + [\omega_{\nu(x)}]) = [\omega(\rho)] \in \omega_*(H_1(F)).$$

Denote by B the intersection form $H_1(\partial M) \times H_1(\partial M) \to \mathbb{Z}$. Set $s_i = [\omega(S_i)] \in H_1(\partial M)$ and $s = s_1 + s_2 + \cdots + s_r \in H_1(\partial M)$. By Section 3.5,

$$\sum_{i=1}^{r} u(\alpha_i) = \sum_{i=1}^{r} \sum_{x \in \bowtie(\omega(S_i)),\, B([\omega_x], s_i) \neq 0} \mathrm{sign}(B([\omega_x], s_i))\, t^{|B([\omega_x], s_i)|}. \tag{5.1.2}$$

For $x \in \bowtie(\omega(S_i))$, the loop ω_x lies on $\omega(S_i)$ and is disjoint from $\cup_{j\neq i}\, \omega(S_j)$. Hence $B([\omega_x], s) = B([\omega_x], s_i)$. Note also that $\bowtie(\omega|_{\partial F}) = \cup_{i=1}^{r} \bowtie(\omega(S_i))$. Therefore Formula 5.1.2 can be rewritten as

$$\sum_{i=1}^{r} u(\alpha_i) = \sum_{x \in \bowtie(\omega|_{\partial F}),\, B([\omega_x], s) \neq 0} \mathrm{sign}(B([\omega_x], s))\, t^{|B([\omega_x], s)|}. \tag{5.1.3}$$

Assume that the genus of F is zero. We shall show that each orbit of the involution ν on $\bowtie(\omega|_{\partial F})$ contributes 0 to the right hand side of Formula 5.1.3. This will imply the claim of the lemma. It suffices to prove that for any $x \in \bowtie(\omega|_{\partial F})$ we have $B([\omega_x], s) = 0$ in the case $\nu(x) = x$ and $B([\omega_x], s) = -B([\omega_{\nu(x)}], s)$ in the case $\nu(x) \neq x$. Set $h_x = [\omega_x] \in H_1(\partial M)$ if $\nu(x) = x$ and $h_x = [\omega_x] + [\omega_{\nu(x)}] \in H_1(\partial M)$ if $\nu(x) \neq x$. As we know, $\mathrm{in}(h_x) \in \omega_*(H_1(F))$. Since the genus of F is 0, the group $H_1(F)$ is generated by the homology classes of the boundary components. Hence there is an integral linear combination $h = h(x)$ of $s_1, ..., s_r \in H_1(\partial M)$ such that $\mathrm{in}(h_x) = \mathrm{in}(h)$. Then $h_x - h \in K$ where K is the kernel of $\mathrm{in} : H_1(\partial M) \to H_1(M)$. The sum $s = s_1 + ... + s_r$ being represented by the 1-cycle $\partial\, \omega(F)$ also lies in K. It is well known that $B(K \times K) = 0$. Hence $B(h_x - h, s) = 0$. Since the curves $\omega(S_1), ..., \omega(S_r)$ are pairwise disjoint and B is skew-symmetric, $B(h, s) = 0$. Therefore $B(h_x, s) = 0$. This implies our claim. □

We can now finish the proof of the theorem. Let α, β be cobordant strings. By assumption, there is an oriented 3-manifold M and a homotopy $\{\omega_t : S^1 \to M\}_{t\in[0,1]}$ such that ω_0, ω_1 are disjoint (generic) closed curves on ∂M realizing α and $\overline{\beta}$, respectively. The homotopy $\{\omega_t\}_t$ defines a map $\omega : S^1 \times [0,1] \to M$. We provide $S^1 \times [0,1]$ with the orientation obtained as the product of the counterclockwise orientation in S^1 and the right-handed orientation in $[0,1]$. Applying Lemma 5.1.5 to ω we obtain that $u(\alpha_0) + u(\alpha_1) = 0$ where α_i is the string underlying the restriction of ω to $S^1 \times i$ where the orientation of $S^1 \times i$ is induced by the one in $S^1 \times [0,1]$. This is the counterclockwise orientation on $S^1 \times 1$ and the opposite one on $S^1 \times 0$. Therefore $\alpha_1 = \overline{\beta}$ and $\alpha_0 = \alpha^-$. Hence $u(\alpha) = -u(\alpha^-) = -u(\alpha_0) = u(\alpha_1) = u(\overline{\beta}) = u(\beta)$. □

Corollary 5.1.6. *The strings $\alpha_{p,q}, \alpha_{p',q'}$ with $p \neq q, p' \neq q'$ are cobordant if and only if $p = p'$ and $q = q'$.*

This follows from the previous theorem and the formula $u(\alpha_{p,q}) = pt^q - qt^p$. The strings $\alpha_{p,q}$ with $p = q$ are all cobordant to each other as will be shown in the next subsection.

Corollary 5.1.7. *For any integers $r_1, ..., r_k \geq 1$, the polynomial $u^{(r_1,...,r_k)}$ is a cobordism invariant of strings.*

Proof. It suffices to prove that if a string α is cobordant to a string β, then $\alpha^{(r)}$ is cobordant to $\beta^{(r)}$ for $r \geq 1$. Let M be a compact oriented 3-manifolds such that $\alpha, \overline{\beta}$ are realized by disjoint closed curves ω, ω' on ∂M homotopic in M. Let $f : S^1 \times [0,1] \to M$ be a homotopy between $\omega = f|_{S^1\times 0}$ and $\omega' = f|_{S^1\times 1}$. By the Poincaré duality, there is a unique $y \in H^1(M;\mathbb{Z})$ such that $y \cap [M] = [f] \in H_2(M, \partial M;\mathbb{Z})$. Let $\tilde{M} \to M$ be the r-fold covering determined by $y (\mathrm{mod}\, r) \in H^1(M;\mathbb{Z}/r\mathbb{Z})$. The mapping f lifts to $\tilde{M}$ and yields a homotopy between $\alpha^{(r)}$ and $(\overline{\beta})^{(r)} = \overline{\beta^{(r)}}$. □

5.2. **Slice strings.** A virtual string cobordant to a trivial string is *slice*. Clearly, a trivial string is slice. Lemma 5.1.1 implies that strings cobordant to a slice string are slice. By Theorem 5.1.2, a string homotopic to a slice string is slice. By the proof of Corollary 5.1.7, all coverings of a slice string are slice.

Theorem 5.1.4 gives obstructions to the sliceness: the polynomial u and the higher polynomials $u^{(r_1,...,r_k)}$ of a slice string are equal to 0. For example, the strings $\alpha_{p,q}$ with $p \neq q$ are not slice.

It is easy to see that a string is slice if and only if it can be realized on a closed surface Σ by a closed curve contractible in an orientable 3-manifold bounded by Σ. Using the gluing of 3-manifolds along 2-disks in the boundary, we obtain that a string that is a product of slice strings is itself slice. Similarly, using the gluing of 3-manifolds along (subsurfaces of) their boundary we obtain the following cancellation: if a product of a string α with a slice string is slice then α is slice.

We outline a construction of slice strings which mimics the well known fact that a sum of a knot with its mirror image is slice. Namely, for any virtual string α, its appropriate product with $\overline{\alpha}^-$ is slice. Indeed, let S be the core circle of α and let $ab \subset S$ be an arc containing all the endpoints of α. Let $(\alpha', S', a'b' \subset S')$ be a disjoint copy of the triple (α, S, ab). Consider the circle $S'' = (\overline{ab \cup a'b'})/a = a', b = b'$ and provide it with the orientation extending the one on ab. The arrows of α and $\overline{\alpha'}$ are attached to $ab \cup a'b'$ and form in this way a virtual string, α'', with core circle S''. It is clear that α'' is a product of α with $\overline{\alpha}^-$. We claim that α'' is slice. To see this, represent α by a closed curve $\omega : S \to \Sigma$ on a surface Σ. The map ω transforms $S - ab$ onto an embedded arc in Σ disjoint from the rest of the curve. Let $D \subset \Sigma$ be a 2-disk such that $D \cap \omega(S) = \omega(S - ab)$ and $\partial D \cap \omega(S) = \{\omega(a), \omega(b)\}$. Consider the 3-manifold $M = (\Sigma - \mathrm{Int}\, D) \times [0, 1]$. The four paths $\omega(ab) \times 0, \omega(ab) \times 1, \omega(a) \times [0, 1], \omega(b) \times [0, 1]$ form a closed curve on ∂M realizing α'' and contractible in M.

The analogy with knot theory suggests the following definition. A string α is *ribbon* if its core circle has an orientation reversing involution j such that for any arrow (a, b) of α the pair $(j(b), j(a))$ is also an arrow of α. Note that such an involution j is topologically equivalent to the complex conjugation on S^1 and, in particular, has two fixed points. The assumptions on α imply that these two points are not endpoints of α. For example, it is obvious that the string α'' constructed in the previous paragraph is ribbon. Another example: the string $\alpha_{p,p}$ with $p \geq 1$ is ribbon. The ribbonness of a string is not a homotopy property: a string obtained from a ribbon string by homotopy moves may be non-ribbon.

The next lemma shows that all ribbon strings are slice.

Lemma 5.2.1. *Ribbon strings are slice.*

Proof. Let S be the core circle of a ribbon string α and let $j : S \to S$ be an orientation reversing involution transforming arrows of α into arrows of α with opposite orientation. Recall the canonical realization $\omega_\alpha : S \to \Sigma_\alpha$ of α. The involution j induces an involution j' on the graph $\Gamma_\alpha = \omega_\alpha(\Sigma_\alpha)$. We extend j' to the disks $\{D_v\}_v$ used to construct Σ_α by $D_v \to D_{j'(v)}$, $(x, y) \mapsto (-y, -x)$ where v runs over the vertices of Γ_α and x, y are the canonical coordinates in these disks, cf. Section 4.1. The resulting involution extends to the ribbons in the obvious way and yields an orientation reversing involution $J : \Sigma_\alpha \to \Sigma_\alpha$ such that $J\omega_\alpha = \omega_\alpha j$. The set $\mathrm{Fix}(J)$ of fixed points of J consists of two disjoint embedded intervals in Σ_α with endpoints on $\partial\Sigma_\alpha$. Consider the cylinder $\Sigma_\alpha \times [0, 1]$ and identify $a \times 0 = J(a) \times 1$ for all $a \in \Sigma_\alpha - \mathrm{Fix}(J)$. For each $a \in \mathrm{Fix}(J)$, contract $a \times [0, 1] \subset \Sigma_\alpha \times [0, 1]$ into a point. This transforms $\Sigma_\alpha \times [0, 1]$ into an oriented 3-manifold M such that $\partial M \supset \Sigma_\alpha$ and ω_α is contractible in M. □

Remark 5.2.2. Not all slice strings are ribbon. To give an example, consider the ribbon string $\alpha_{1,1}$. Since $\alpha_{1,1}$ is slice, any string obtained as a product of ≥ 2 copies of $\alpha_{1,1}$ is slice. Some of such products are not ribbon. For example, consider the permutation $\sigma = (12)(34)$ on the set $\{1, 2, 3, 4\}$ permuting 1 with 2 and 3 with 4 and consider the rank 4 string α_σ defined in Section 3.3.2. Drawing a picture, one observes that α_σ is a product of two copies of $\alpha_{1,1}$ and is ribbon. Inverting orientation of any arrow of α_σ we obtain a string which is also a product of two copies of $\alpha_{1,1}$ but which is not ribbon by obvious geometric reasons.

5.3. **Slice genus.** The *slice genus* $sg(\alpha)$ of a string α is the minimal integer $k \geq 0$ satisfying the following condition: there are an oriented 3-manifold M, a compact (oriented) surface F of genus k bounded by a circle, and a proper map $\omega : F \to M$ such that $\omega|_{\partial F} : \partial F \to \partial M$ is a (generic) closed curve on ∂M realizing α. (The word "proper" means that $\omega(\partial F) \subset \partial M$.) Such k exists because any loop on a closed surface is homologically trivial in a certain handlebody bounded by this surface. It is clear that $sg(\alpha)$ is a cobordism invariant of α. A string α is slice if and only if $sg(\alpha) = 0$.

We similarly define a slice genus for tuples of strings. The *slice genus* $sg(\alpha_1, ..., \alpha_r)$ of $r \geq 1$ strings $\alpha_1, ..., \alpha_r$ is the minimal integer $k \geq 0$ satisfying the following condition: there are an oriented 3-manifold M, a compact (oriented) surface F of genus k bounded by r circles $S_1, ..., S_r$, and a proper map $\omega : F \to M$ such that the maps $\omega|_{S_i} : S_i \to \partial M$ with $i = 1, ..., r$ are disjoint (generic) closed curves on ∂M realizing $\alpha_1, ..., \alpha_r$, respectively. The existence of such k can be obtained by realizing $\alpha_1, ..., \alpha_r$ by curves on disjoint surfaces, taking the connected sum of these surfaces and presenting the result as a boundary of an appropriate

handlebody. We do not require M or F to be connected although it is always possible to achieve their connectedness by taking connected sum. Note that the genus of a disconnected surface is by definition the sum of the genera of its components.

Clearly $sg(\alpha_1, ..., \alpha_r) \geq 0$. If $sg(\alpha_1, ..., \alpha_r) = 0$ then we call the sequence $\alpha_1, ..., \alpha_r$ *slice*. The same argument as in the proof of Corollary 5.1.7 shows that if $\alpha_1, ..., \alpha_r$ is slice, then for any integer $m \geq 1$, the sequence of the m-th coverings $\alpha_1^{(m)}, ..., \alpha_r^{(m)}$ is slice. By Lemma 5.1.5, if $\alpha_1, ..., \alpha_r$ is slice, then for any finite sequence of positive integers $m_1, ..., m_k$ we have $u^{(m_1,...,m_k)}(\alpha_1) + ... + u^{(m_1,...,m_k)}(\alpha_r) = 0$.

The number $sg(\alpha_1, ..., \alpha_r)$ does not depend on the order in the tuple $\alpha_1, ..., \alpha_r$. This number is preserved if we replace $\alpha_1, ..., \alpha_r$ with cobordant strings. If α_r is slice, then $sg(\alpha_1, ..., \alpha_r) = sg(\alpha_1, ..., \alpha_{r-1})$. Reversing orientations in 3-manifolds and/or surfaces F, we obtain

$$sg(\overline{\alpha}_1, ..., \overline{\alpha}_r) = sg(\alpha_1^-, ..., \alpha_r^-) = sg(\alpha_1, ..., \alpha_r).$$

Using the gluing as in the proof of Lemma 5.1.1, we obtain that for any $0 \leq s \leq r$ and any strings $\alpha, \alpha_1, ..., \alpha_r$,

$$sg(\alpha_1, ..., \alpha_r) \leq sg(\alpha_1, ..., \alpha_s, \alpha) + sg(\overline{\alpha}^-, \alpha_{s+1}, ..., \alpha_r).$$

When α is a trivial string, this gives the obvious inequality $sg(\alpha_1, ..., \alpha_r) \leq sg(\alpha_1, ..., \alpha_s) + sg(\alpha_{s+1}, ..., \alpha_r)$.

For $r = 2$, we can rewrite the slice genus in the equivalent form $sg'(\alpha, \beta) = sg(\alpha, \overline{\beta}^-) = sg(\alpha^-, \overline{\beta})$ for strings α, β. The results of the previous paragraph imply that the number $sg'(\alpha, \beta)$ depends only on the cobordism classes of α, β and defines a metric on the set of cobordism classes of strings: $sg'(\alpha, \beta) = 0$ if and only if α and β are cobordant (cf. the end of the proof of Theorem 5.1.4), $sg'(\alpha, \beta) = sg'(\beta, \alpha)$, and $sg'(\alpha, \beta) \leq sg'(\alpha, \gamma) + sg'(\gamma, \beta)$ for any strings α, β, γ. Note also that $sg(\alpha) = sg'(\alpha, O)$ where O is a trivial string and $sg'(\alpha, \beta) \leq sg(\alpha) + sg(\beta)$.

5.4. **Adams operations on strings.** We can define "Adams operations" $\{\psi^n\}_{n \in \mathbb{Z}}$ on the set of homotopy classes of strings. Let α be a virtual string. Replacing α by a homeomorphic string, we can identify its core circle with $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Consider a curve $\omega : S^1 \to \Sigma$ realizing α on a surface Σ. The mapping $S^1 \to \Sigma$ sending $z \in S^1$ to $\omega(z^n)$ is homotopic to a generic curve $S^1 \to \Sigma$. We define $\psi^n(\alpha)$ to be the homotopy class of its underlying string. Lemma 5.1.3 implies that $\psi^n(\alpha)$ depends neither on the choice of ω nor on the choice of α in its homotopy class. Clearly, $\psi^1(\alpha) = \alpha$, $\psi^{-1}(\alpha) = \alpha^-$, and $\psi^{mn} = \psi^m \circ \psi^n$ for any $m, n \in \mathbb{Z}$. It is also clear that ψ^n transforms cobordant strings into cobordant strings and induces thus an "Adams operation" on the set of cobordism classes of strings. As an exercise, the reader may check that $u(\psi^n(\alpha)) = \mathrm{sign}(n)\, n^2\, u(\alpha)$.

5.5. **Remarks.** 1. J. S. Carter [Ca2] first observed that there are closed curves on surfaces that bound no singular disks in 3-manifolds bounded by these surfaces. One of his results can be rephrased by saying that the string $\alpha_{2,1}$ is not slice. Carter's technique consists in studying certain partitions (called filamentations) of the set of double points of a curve into pairs and singletons. In the notation of the proof of Lemma 5.1.5 (where F should be a disk), a filamentation is formed by the orbits of the involution μ. Carter's obstruction to the sliceness is formulated in terms of intersection numbers of the intervals of double points on the disk. Note also a relevant result of [HK] (Theorem 4.10): if a closed curve on a surface has a filamentation then any homotopic closed curve also has a filamentation.

2. The definition of cobordism for strings requires only the *existence* of disjoint realizations homotopic in a 3-manifold. Note that *any* two disjoint curves realizing cobordant strings on a closed surface are homotopic in a certain oriented 3-manifold bounded by this surface. To see this, one needs the following two observations.

(i) For any disjoint realizations ω_1, ω_2 of strings α_1, α_2 on a closed surface Σ, there are realizations ω_1', ω_2' of α_1, α_2 on disjoint closed surfaces Σ_1, Σ_2 and an oriented 3-manifold M with $\partial M = \Sigma \cup (-\Sigma_1) \cup (-\Sigma_2)$ such that ω_i is homotopic to ω_i' in M for $i = 1, 2$. This can be proven by adding 2-handles along the simple closed curves in Σ bounding a regular neighborhood of ω_1 in Σ.

(ii) For any realizations ω, ω' of the same string on closed surfaces Σ, Σ', there is an oriented 3-manifold M with $\partial M = \Sigma \cup (-\Sigma')$ such that ω is homotopic to ω' in M. This can be deduced from the fact that both ω, ω' can be obtained from the canonical realization of the string on a closed surface of minimal genus by adding 1-handles.

6. Based matrices of strings

6.1. **Based matrices.** Fix an abelian group H. A *based skew-symmetric matrix over* H or shortly a *based matrix* is a triple $(G, s, b : G^2 = G \times G \to H)$ where G is a finite set, $s \in G$, and the mapping b is skew-symmetric in the sense that $b(g,h) = -b(h,g)$ for all $g, h \in G$ and $b(g,g) = 0$ for all $g \in G$.

We call an element $g \in G - \{s\}$ *annihilating* (with respect to b) if $b(g,h) = 0$ for all $h \in G$. We call $g \in G - \{s\}$ a *core element* if $b(g,h) = b(s,h)$ for all $h \in G$. We call two elements $g_1, g_2 \in G - \{s\}$ *complementary* if $b(g_1,h) + b(g_2,h) = b(s,h)$ for all $h \in G$. A based matrix (G, s, b) is *primitive* if it has no annihilating elements, no core elements, and no complementary pairs of elements. An example of a primitive based matrix is provided by the *trivial based matrix* (G, s, b) where G consists of only one element s and $b(s,s) = 0$.

We define three operations M_1, M_2, M_3 on based matrices, called *elementary extensions*. They add to a based matrix (G, s, b) an annihilating element, a core element, and a pair of complementary elements, respectively. More precisely, M_1 transforms (G, s, b) into the (unique) based matrix $(\overline{G} = G \amalg \{g\}, s, \overline{b})$ such that $\overline{b} : \overline{G} \times \overline{G} \to H$ extends b and $\overline{b}(g,h) = 0$ for all $h \in \overline{G}$. The move M_2 transforms (G, s, b) into the (unique) based matrix $(\tilde{G} = G \amalg \{g\}, s, \tilde{b})$ such that $\tilde{b} : \tilde{G} \times \tilde{G} \to H$ extends b and $\tilde{b}(g,h) = \tilde{b}(s,h)$ for all $h \in \tilde{G}$. The move M_3 transforms (G, s, b) into a based matrix $(\hat{G} = G \amalg \{g_1, g_2\}, s, \hat{b})$ where $\hat{b} : \hat{G} \times \hat{G} \to H$ is any skew-symmetric map extending b and such that $\hat{b}(g_1,h) + \hat{b}(g_2,h) = \hat{b}(s,h)$ for all $h \in \hat{G}$. It is clear that a based matrix is primitive if and only if it cannot be obtained from another based matrix by an elementary extension.

Two based matrices (G, s, b) and (G', s', b') are *isomorphic* if there is a bijection $G \to G'$ sending s into s' and transforming b into b'. To specify the isomorphism class of a based matrix (G, s, b), it suffices to specify the matrix $(b(g,h))_{g,h \in G}$ where it is understood that the first column and row correspond to s. In this way every skew-symmetric square matrix over H (with zeroes on the diagonal) determines a based matrix.

Two based matrices are *homologous* if one can be obtained from the other by a finite sequence of elementary extensions M_1, M_2, M_3, the inverse transformations, and isomorphisms. The homology is an equivalence relation on the set of based matrices.

Lemma 6.1.1. *Every based matrix is obtained from a primitive based matrix by elementary extensions. Two homologous primitive based matrices are isomorphic.*

Proof. The first claim is obvious: eliminating annihilating elements, core elements, and complementary pairs of elements by the moves M_i^{-1} with $i = 1, 2, 3$ we can transform any based matrix T into a primitive based matrix $T_\bullet$. Then T is obtained from $T_\bullet$ by elementary extensions.

To prove the second claim, we need the following assertion:

$(*)$ a move M_i followed by M_j^{-1} yields the same result as an isomorphism, or a move $M_k^{\pm 1}$, or a move M_k^{-1} followed by M_l with $k, l \in \{1, 2, 3\}$.

This assertion will imply the second claim of the lemma. Indeed, suppose that two primitive based matrices T, T' are related by a finite sequence of transformations $M_1^{\pm 1}, M_2^{\pm 1}, M_3^{\pm 1}$ and isomorphisms. An isomorphism of based matrices followed by $M_i^{\pm 1}$ can be also obtained as $M_i^{\pm 1}$ followed by an isomorphism. Therefore all isomorphisms in our sequence can be accumulated at the end. The claim $(*)$ implies that T, T' can be related by a finite sequence of moves consisting of several moves of type M_i^{-1} followed by several moves of type M_i and isomorphisms. However, since T is primitive we cannot apply to it a move of type M_i^{-1}. Hence there are no such moves in our sequence. Similarly, since T' (and any isomorphic based matrix) is primitive, it cannot be obtained by an application of M_i. Therefore our sequence consists solely of isomorphisms so that T is isomorphic to T'.

Let us now prove $(*)$. We have to consider nine cases depending on $i, j \in \{1, 2, 3\}$.

For $i, j \in \{1, 2\}$, the move M_i on a based matrix (G, s, b) adds one element g and then M_j^{-1} removes one element $g' \in G \amalg \{g\}$. If $g' = g$, then $M_j^{-1} \circ M_i$ is the identity. If $g' \neq g$, then $g' \in G$ is annihilating (resp. core) for $j = 1$ (resp. $j = 2$). The transformation $M_j^{-1} \circ M_i$ can be achieved by first applying M_j^{-1} that removes g' and then applying M_i that adds g.

Let $i = 1, j = 3$. The move M_i on (G, s, b) adds an annihilating element g and M_j^{-1} removes two complementary elements $g_1, g_2 \in G \amalg \{g\}$. If $g_1 \neq g$ and $g_2 \neq g$, then $g_1, g_2 \in G$ and $M_j^{-1} \circ M_i$ can be achieved by first removing g_1, g_2 and then adding g. If $g_1 = g$, then g_2 is a core element of G and $M_j^{-1} \circ M_i$ is the move M_2^{-1} removing g_2. The case $g_2 = g$ is similar.

Let $i=2, j=3$. The move M_i on (G,s,b) adds a core element g and M_j^{-1} removes two complementary elements $g_1, g_2 \in G \amalg \{g\}$. If $g_1 \neq g$ and $g_2 \neq g$, then $M_j^{-1} \circ M_i$ can be achieved by first removing g_1, g_2 and then adding g. If $g_1 = g$, then $g_2 \in G$ is an annihilating element of G and $M_j^{-1} \circ M_i$ is the move M_1^{-1} removing g_2. The case $g_2 = g$ is similar.

Let $i=3, j=1$. The move M_i on (G,s,b) adds two complementary elements g_1, g_2 and M_j^{-1} removes an annihilating element $g \in G \amalg \{g_1, g_2\}$. If $g \neq g_1$ and $g \neq g_2$, then $g \in G$ and $M_j^{-1} \circ M_i$ can be achieved by first removing g and then adding g_1, g_2. If $g = g_1$, then g_2 is a core element of $G \amalg \{g_2\}$ and $M_j^{-1} \circ M_i = M_2$. The case $g = g_2$ is similar.

Let $i=3, j=2$. The move M_i on (G,s,b) adds two complementary elements g_1, g_2 and M_j^{-1} removes a core element $g \in G \amalg \{g_1, g_2\}$. If $g \neq g_1$ and $g \neq g_2$, then $g \in G$ and $M_j^{-1} \circ M_i$ can be achieved by first removing g and then adding g_1, g_2. If $g = g_1$, then g_2 is an annihilating element of $G \amalg \{g_2\}$ and $M_j^{-1} \circ M_i = M_1$. The case $g = g_2$ is similar.

Let $i=j=3$. The move M_i on (G,s,b) adds two complementary elements g_1, g_2 and M_j^{-1} removes two complementary elements $g_1', g_2' \in G \amalg \{g_1, g_2\}$. If these two pairs are disjoint, then $M_j^{-1} \circ M_i$ can be achieved by first removing $g_1', g_2' \in G$ and then adding g_1, g_2. If these two pairs coincide, then $M_j^{-1} \circ M_i$ is the identity. It remains to consider the case where these pairs have one common element, say $g_1' = g_1$, while $g_2' \neq g_2$. Then $g_2' \in G$ and for all $h \in G$,

$$\hat{b}(g_2, h) = \hat{b}(s,h) - \hat{b}(g_1, h) = \hat{b}(s,h) - \hat{b}(g_1', h) = \hat{b}(g_2', h) = b(g_2', h).$$

Therefore the move $M_j^{-1} \circ M_i$ gives a based matrix isomorphic to (G,s,b). The isomorphism $G \to (G - \{g_2'\}) \cup \{g_2\}$ is the identity on $G - \{g_2'\}$ and sends g_2' into g_2. □

Lemma 6.1.1 implies that each based matrix $T=(G,s,b)$ is homologous to a primitive based matrix $T_\bullet = (G_\bullet, s_\bullet, b_\bullet)$ unique up to isomorphism. This reduces classification of based matrices up to homology to a classification of primitive based matrices up to isomorphism. Note that we can choose $T_\bullet$ in its isomorphism class so that $G_\bullet \subset G$ and $b_\bullet$ is the restriction of b to $G_\bullet \times G_\bullet$.

We define two more operations on based matrices. For a based matrix $T=(G,s,b)$, set $-T = (G,s,-b)$ and $T^- = (G,s,b^-)$ where $b^-(s,h) = -b(s,h)$, $b^-(h,s) = -b(h,s)$ for all $h \in G$ and $b^-(g,h) = b(g,h) - b(g,s) - b(s,h)$ for all $g,h \in G - \{s\}$. The transformations $T \mapsto -T$, $T \mapsto T^-$ are commuting involutions on the set of based matrices. It is easy to check that they are compatible with homology and preserve the class of primitive based matrices. It follows from the definitions that $(-T)_\bullet = -T_\bullet$ and $(T^-)_\bullet = (T_\bullet)^-$.

Remarks 6.1.2. 1. The moves M_1, M_2, M_3 on based matrices are not independent. It is easy to present M_2 as a composition of M_3 with M_1^{-1}.

2. Each isomorphism invariant v of primitive based matrices extends to a homology invariant of based matrices by $v(T) = v(T_\bullet)$. The most important numerical invariant of a primitive based matrix (G,s,b) is the number $\#(G)$. It is easy to define further invariants of primitive based matrices. For instance, for $k \in H$, we can set

$$v_k(G,s,b) = \#\{g \in G \,|\, b(g,s) = k\}.$$

Similarly, for $k \in H$ and a finite set A of elements of H endowed with non-negative multiplicities, set

$$v_{k,A}(G,s,b) = \#\{g \in G \,|\, b(g,s) = k \text{ and } \{b(g,h)\}_{h \in G - \{s\}} = A\},$$

where the latter equality is understood as an equality of sets with multiplicities. Clearly, $v_k = \sum_A v_{k,A}$.

3. If $H \subset \mathbb{R}$ is a subgroup of the additive group of real numbers, then the 1-variable polynomial

$$u(T)\,(t) = \sum_{g \in G, b(g,s) \neq 0} \operatorname{sign}(b(g,s))\, t^{|b(g,s)|}$$

is a homology invariant of a based matrix $T=(G,s,b)$.

6.2. **The based matrix of a string.** With each virtual string α we associate a based matrix $T(\alpha) = (G,s,b)$ over $\mathbb{Z}$. Set $G = G(\alpha) = \{s\} \amalg \operatorname{arr}(\alpha)$. To define $b = b(\alpha) : G \times G \to \mathbb{Z}$, we identify G with the basis $s \cup \{[e]\}_{e \in \operatorname{arr}(\alpha)}$ of $H_1(\Sigma_\alpha)$, see Section 4.2. The map b is obtained by restricting the homological intersection pairing $H_1(\Sigma_\alpha) \times H_1(\Sigma_\alpha) \to \mathbb{Z}$ to G. It is clear that b is skew-symmetric. We can compute b combinatorially using Formula 3.5.1 and Lemma 4.2.1. In particular, $b(e,s) = n(e)$ for all $e \in \operatorname{arr}(\alpha)$.

The map b can be computed from any closed curve ω realizing α on a surface Σ. Indeed, such a curve is obtained from the canonical realization of α in Σ_α via an orientation-preserving embedding $\Sigma_\alpha \hookrightarrow \Sigma$. It remains to observe that such an embedding preserves intersection numbers and transforms the basis $s \cup \{[e]\}_{e\in \mathrm{arr}(\alpha)}$ of $H_1(\Sigma_\alpha)$ into the subset $[\omega], \{[\omega_x]\}_{x\in\bowtie(\omega)}$ of $H_1(\Sigma)$, cf. Section 3.5.

Lemma 6.2.1. *If two virtual strings are homotopic, then their based matrices are homologous.*

Proof. By Lemma 5.1.3 it is enough to show that if two closed curves ω, ω' on a surface Σ are homotopic, then the based matrices of their underlying strings are homologous. By the discussion in Section 2.3, it suffices to consider the case where ω' is obtained from ω by one of the local moves listed there.

If ω' is obtained from ω by adding a small curl, then $\bowtie(\omega') = \bowtie(\omega) \cup \{y\}$ where y is a new crossing. Clearly $[\omega'_y] = 0 \in H_1(\Sigma)$ or $[\omega'_y] = [\omega'] = [\omega] \in H_1(\Sigma)$ depending on whether the curl lies on the right or on the left of ω. Also $[\omega'_x] = [\omega_x]$ for all $x \in \bowtie(\omega)$. Hence $T(\beta)$ is obtained from $T(\alpha)$ by M_1 or M_2.

Suppose that ω' is obtained from ω by the move pushing a branch of ω across another branch and creating two new double points y, z. Clearly, $[\omega'_x] = [\omega_x]$ for all $x \in \bowtie(\omega) \subset \bowtie(\omega')$. It is easy to see that $[\omega'_y] + [\omega'_z] = [\omega'] = [\omega] \in H_1(\Sigma)$. Therefore $T(\beta)$ is obtained from $T(\alpha)$ by M_3.

If ω' is obtained from ω by pushing a branch of ω across a double point, then the subsets $[\omega], \{[\omega_x]\}_{x\in\bowtie(\omega)}$ and $[\omega'], \{[\omega'_x]\}_{x\in\bowtie(\omega')}$ of $H_1(\Sigma)$ coincide so that $T(\alpha)$ is isomorphic to $T(\beta)$. □

6.3. **Invariants of strings from based matrices.** Every virtual string α gives rise to a primitive based matrix $T_\bullet(\alpha)$ over $\mathbb{Z}$ by $T_\bullet(\alpha) = (T(\alpha))_\bullet$. This is the only primitive based matrix (up to isomorphism) homologous to $T(\alpha)$. By Lemma 6.2.1, the based matrix $T_\bullet(\alpha) = (G_\bullet, s_\bullet, b_\bullet)$ is a homotopy invariant of α. This based matrix determines the polynomial $u(\alpha)$ introduced in Section 3: it follows from Formulas 3.2.2 and 3.5.1 that $u(\alpha) = u(T(\alpha)) = u(T_\bullet(\alpha))$. The number $\rho(\alpha) = \#(G_\bullet) - 1$ is a useful homotopy invariant of α which may be non-zero even when $u(\alpha) = 0$, cf. the examples below. Note that if α is homotopically trivial, then $T_\bullet(\alpha)$ is a trivial based matrix and $\rho(\alpha) = 0$.

It follows from the definitions that $T(\alpha^-) = (T(\alpha))^-$ and therefore $T_\bullet(\alpha^-) = (T_\bullet(\alpha))^-$. Similarly, $T(\overline{\alpha}) = -(T(\alpha))^-$ and $T_\bullet(\overline{\alpha}) = -(T_\bullet(\alpha))^-$.

The based matrix $T_\bullet(\alpha) = (G_\bullet, s_\bullet, b_\bullet)$ can be used to estimate the homotopy rank and the homotopy genus of α. Namely, $hr(\alpha) \geq \rho(\alpha)$ since any string homotopic to α must have at least $\rho(\alpha)$ arrows. Similarly, $hg(\alpha) \geq (1/2)\,\mathrm{rank}\, b_\bullet$ where $\mathrm{rank}\, b_\bullet$ is the rank of the integral matrix $(b_\bullet(g,h))_{g,h\in G_\bullet}$. Indeed, if α' is a string homotopic to α and $T(\alpha') = (G', s', b')$, then $g(\alpha') = (1/2)\,\mathrm{rank}\, b' \geq (1/2)\,\mathrm{rank}\, b_\bullet$ since the matrix of b' contains the matrix of $b_\bullet$ as a submatrix.

Combining the inequalities $hr(\alpha) \geq \rho(\alpha)$, $hg(\alpha) \geq (1/2)\,\mathrm{rank}\, b_\bullet$ with the obvious inequalities $\mathrm{rank}\,\alpha \geq hr(\alpha)$ and $g(\alpha) \geq hg(\alpha)$, we obtain that if $T(\alpha)$ is primitive, then $hr(\alpha) = \mathrm{rank}\,\alpha$ and $hg(\alpha) = g(\alpha)$.

6.4. **Applications.** (1) The based matrix $T(\alpha_{p,q})$ of the string $\alpha_{p,q}$ with $p, q \geq 1$ was computed in Section 4.3. It is easy to check that except in the case $p = q = 1$, this based matrix is primitive. Thus $T_\bullet(\alpha_{p,q}) = T(\alpha_{p,q})$, $hr(\alpha_{p,q}) = \mathrm{rank}\,\alpha_{p,q} = p + q$ and $hg(\alpha_{p,q}) = g(\alpha_{p,q})$ provided $p \neq 1$ or $q \neq 1$. In particular, $\alpha_{p,p}$ is a homotopically non-trivial string with zero u-polynomial for all $p > 1$.

(2) The product of strings defined in Section 2.4 does not induce a well-defined operation on the set of homotopy classes of strings. To see this, we exhibit a homotopically non-trivial string which is a product of two copies of a homotopically trivial string. Namely, the string α_σ considered in Section 5.2.2 has the required properties. It is observed there that α_σ is a product of two copies of the homotopically trivial string $\alpha_{1,1}$. The based matrix $T(\alpha_\sigma)$ can be explicitly computed, cf. Section 4.3.2. It is determined by the following skew-symmetric matrix over $\mathbb{Z}$:

$$\begin{bmatrix} 0 & -1 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 & 1 \\ -1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

It is easy to check that this based matrix is primitive. Hence α_σ is not homotopically trivial. Moreover, it is not homotopic to a string with < 4 arrows.

(3) We prove that the involution $\alpha \mapsto \overline{\alpha}$ acts non-trivially on the set of homotopy classes of strings. Consider the permutation $\sigma = (134)(2)$ on the set $\{1,2,3,4\}$ sending 1 to 3, 3 to 4, 4 to 1, and 2 to 2. Drawing the string α_σ we obtain that $\overline{\alpha}_\sigma = \alpha_\tau$ where τ is the permutation $(124)(3)$. The based matrices

$T(\alpha_\sigma)$ and $T(\alpha_\tau)$ can be explicitly computed. They are determined by the following skew-symmetric matrices:

$$\begin{bmatrix} 0 & -2 & 0 & -1 & 3 \\ 2 & 0 & 1 & 0 & 3 \\ 0 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ -3 & -3 & -2 & -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1 & -2 & 0 & 3 \\ 1 & 0 & -1 & 1 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & 0 & 1 \\ -3 & -3 & -2 & -1 & 0 \end{bmatrix}.$$

The based matrices $T(\alpha_\sigma)$ and $T(\alpha_\tau)$ are not isomorphic; this is clear for instance from the fact that the first matrix has a row with three zeros while the second matrix does not have such a row. It is clear also that these based matrices are primitive. By Lemma 6.1.1, they are not homologous. Hence α_σ is not homotopic to $\alpha_\tau = \overline{\alpha}_\sigma$.

7. Genus and cobordism for based matrices

Throughout this section the symbol R denotes a domain, i.e., a commutative ring with unit and with no zero-divisors. By a based matrix over R, we mean a based matrix over the additive group of R.

7.1. **Genus of based matrices.** We define a numerical invariant of a based matrix $T = (G, s, b)$ over R called its *genus* and denoted $\sigma(T)$. For subsets $X, Y \subset G$, set $b(X,Y) = \sum_{g\in X, h\in Y} b(g,h) \in R$. Clearly, $b(X,Y) = -b(Y,X)$ and $b(X,X) = b(\emptyset, X) = 0$ for all $X, Y \subset G$. A *(simple) filling* $\mathcal{X}$ of T is a finite family $\{X_i\}_i$ of disjoint (possibly empty) subsets of G such that $\cup_i X_i = G$, $\#(X_i) \leq 2$ for all i, and one of X_i is the one-element set $\{s\}$. The *matrix* of $\mathcal{X} = \{X_i\}_i$ is the matrix $(b(X_i, X_j))_{i,j}$. This is a skew-symmetric square matrix (with zero diagonal) over R. Its rank (the maximal size of a non-zero minor) is an even non-negative integer; let $\sigma(\mathcal{X})$ denote half of this rank. Set $\sigma(T) = \min_{\mathcal{X}} \sigma(\mathcal{X})$ where $\mathcal{X}$ runs over all fillings of T. Extending b by linearity to the R-module $\Lambda = RG$ freely generated by G and identifying a subset $X \subset G$ with the vector $\sum_{g\in X} g \in \Lambda$, we can interpret $\sigma(T)$ as half the minimal rank of the restriction of b to the submodules of Λ arising from fillings of T.

Note that $\sigma(T) \geq 0$ and $\sigma(T) = 0$ if and only if T has a filling with zero matrix. In the latter case we say that T is *hyperbolic*.

Lemma 7.1.1. *The genus of a based matrix is a homology invariant.*

Proof. By Remark 6.1.2.1, it suffices to prove that $\sigma(T) = \sigma(T')$ for any based matrix $T' = (G', s, b')$ obtained from a based matrix $T = (G, s, b)$ by a move M_i with $i = 1, 3$. The set $G' - G$ consists of one element if $i = 1$ and of two elements if $i = 3$. Pick a filling $\mathcal{X} = \{X_i\}_i$ of T such that $\sigma(T) = \sigma(\mathcal{X})$. Consider the filling $\mathcal{X}' = (G' - G) \cup \{X_i\}_i$ of T'. Its matrix is obtained from the one of $\mathcal{X}$ by adjoining a row and a column. For $i = 1$, these row and column are zero so that $\sigma(\mathcal{X}') = \sigma(\mathcal{X})$. For $i = 3$, we have $b(G' - G, Y) = b(\{s\}, Y)$ for all $Y \subset G$. Since one of the sets X_i equals $\{s\}$, we again obtain $\sigma(\mathcal{X}') = \sigma(\mathcal{X})$. Hence $\sigma(T') \leq \sigma(\mathcal{X}') = \sigma(\mathcal{X}) = \sigma(T)$.

To prove the opposite inequality, pick a filling $\mathcal{X}' = \{X_i\}_i$ of T' such that $\sigma(T') = \sigma(\mathcal{X}')$. We shall construct a filling $\mathcal{X}$ of T such that $\sigma(\mathcal{X}) \leq \sigma(\mathcal{X}')$. This would imply $\sigma(T) \leq \sigma(\mathcal{X}) \leq \sigma(\mathcal{X}') = \sigma(T')$. Consider the case $i = 1$. One of the sets X_i contains the 1-element set $G' - G$. We replace this X_i by $X_i - (G' - G)$ and keep all the other X_i. This gives a filling $\mathcal{X}$ of T whose matrix coincides with the matrix of $\mathcal{X}'$. Hence $\sigma(\mathcal{X}) = \sigma(\mathcal{X}')$. Let now $i = 3$. If one of the sets X_i is equal to $G' - G = \{g_1, g_2\}$, then removing this X_i from $\mathcal{X}'$ we obtain a filling $\mathcal{X}$ of T. As in the previous paragraph, $\sigma(\mathcal{X}) = \sigma(\mathcal{X}')$. Suppose that the elements g_1, g_2 of $G' - G$ belong to different subsets, say X_1, X_2, of the filling $\mathcal{X}'$. Then the sets X_i with $i \neq 1, 2$ and $X = (X_1 \cup X_2) - \{g_1, g_2\}$ form a filling of T. Let X_0 be the term of the fillings $\mathcal{X}$ and $\mathcal{X}'$ equal to $\{s\}$. For any $Y \subset G$,

$$b(X,Y) = b'(X,Y) = b'(X_1,Y) + b'(X_2,Y) - b'(g_1,Y) - b'(g_2,Y) = b'(X_1,Y) + b'(X_2,Y) - b'(X_0,Y).$$

Applying this to $Y = X_i$ with $i \neq 1, 2$, we obtain that the skew-symmetric bilinear form determined by the matrix of $\mathcal{X}$ is induced from the skew-symmetric bilinear form determined by the matrix of $\mathcal{X}'$ via the linear map of the corresponding free R-modules sending the basis vectors X and $\{X_i\}_{i\neq 1,2}$ respectively to $X_1 + X_2 - X_0$ and $\{X_i\}_{i\neq 1,2}$. Hence $\sigma(\mathcal{X}) \leq \sigma(\mathcal{X}')$. □

Corollary 7.1.2. *For any based matrix T over R, we have $\sigma(T_\bullet) = \sigma(T)$. A based matrix over R homologous to a hyperbolic based matrix is itself hyperbolic.*

7.2. **Genus for tuples of based matrices.** The definition of the genus of a based matrix can be extended to tuples of based matrices. Consider a tuple of $r \geq 1$ based matrices $T_1 = (G_1, s_1, b_1), ..., T_r = (G_r, s_r, b_r)$ over R. Replacing $T_1, ..., T_r$ by isomorphic based matrices, we can assume that the sets $G_1, ..., G_r$ are disjoint. Let $\Lambda = RG$ be the free R-module with basis $G = \cup_{t=1}^r G_t$. Let Λ_s be the submodule of Λ generated by $s_1, ..., s_r$. We call a vector $x \in \Lambda$ *short* if $x \in \Lambda_s$ or $x \in g + \Lambda_s$ for some $g \in G - \{s_1, ..., s_r\}$ or $x \in g + h + \Lambda_s$ for distinct $g, h \in G - \{s_1, ..., s_r\}$. A *filling* of $T_1, ..., T_r$ is a finite family $\{\lambda_i\}_i$ of short vectors in Λ such that $\sum_i \lambda_i = \sum_{g \in G} g (\operatorname{mod} \Lambda_s)$ and one of λ_i is equal to $s_1 + s_2 + ... + s_r$. Note that each element of $G - \{s_1, ..., s_r\}$ appears in exactly one λ_i with non-zero coefficient; this coefficient is then $+1$. The basis vectors $s_1, ..., s_r$ may appear in several λ_i with non-zero coefficients.

The maps $\{b_t : G_t \times G_t \to R\}_t$ induce a skew-symmetric bilinear form $b = \oplus_t b_t : \Lambda \times \Lambda \to R$ such that $b(g, h) = b_t(g, h)$ for $g, h \in G_t$ and $b(G_t, G_{t'}) = 0$ for $t \neq t'$. The *matrix* of a filling $\lambda = \{\lambda_i\}_i$ of $T_1, ..., T_r$ is the matrix $(b(\lambda_i, \lambda_j))_{i,j}$. This is a skew-symmetric square matrix over R. Let $\sigma(\lambda) \in \mathbb{Z}$ be half of its rank. Set $\sigma(T_1, ..., T_r) = \min_\lambda \sigma(\lambda)$ where λ runs over all fillings of $T_1, ..., T_r$. Clearly $\sigma(T_1, ..., T_r) \geq 0$ and $\sigma(T_1, ..., T_r) = 0$ if and only if $(T_1, ..., T_r)$ has a filling with zero matrix. In the latter case we call the sequence $T_1, ..., T_r$ *hyperbolic*.

It is obvious that the genus $\sigma(T_1, ..., T_r)$ is preserved when $T_1, ..., T_r$ are permuted or replaced with isomorphic based matrices. Also $\sigma(-T_1, ..., -T_r) = \sigma(T_1, ..., T_r)$. If T_r is a trivial based matrix, then $\sigma(T_1, ..., T_r) = \sigma(T_1, ..., T_{r-1})$ (because then the vector $s_r \in \Lambda$ lies in the annihilator of b).

For $r = 1$, the notion of a filling is slightly wider than the notion of a simple filling in Section 7.1. However, they give the same genus and the same set of hyperbolic based matrices.

Lemma 7.2.1. *For any $1 \leq t \leq r$ and any based matrices $T_0, T_1, ..., T_r$,*

$$\sigma(T_1, ..., T_r) \leq \sigma(T_1, ..., T_t, T_0) + \sigma(-T_0, T_{t+1}, ..., T_r).$$

Proof. Consider for concreteness the case where $t = 1$ and $r = 2$, the general case is quite similar. We must prove that $\sigma(T_1, T_2) \leq \sigma(T_1, T_0) + \sigma(-T_0, T_2)$. Let $T_i = (G_i, s_i, b_i)$ for $i = 0, 1, 2$ and $T'_0 = (G'_0, s'_0, b'_0)$ be a copy of T_0 where $G'_0 = \{g' \,|\, g \in G_0\}$, $s'_0 = (s_0)'$, and b'_0 is defined by $b'_0(g', h') = b_0(g, h)$ for $g, h \in G_0$. We can assume that the sets G_1, G_0, G'_0, G_2 are disjoint. Let $\Lambda_1, \Lambda_0, \Lambda'_0, \Lambda_2$ be free R-modules freely generated by G_1, G_0, G'_0, G_2, respectively, and let $\Lambda = \Lambda_1 \oplus \Lambda_0 \oplus \Lambda'_0 \oplus \Lambda_2$. There is a unique skew-symmetric bilinear form $B = b_1 \oplus b_0 \oplus (-b'_0) \oplus b_2$ on Λ such that the sets $G_1, G_0, G'_0, G_2 \subset \Lambda$ are mutually orthogonal and the restrictions of B to these subsets are equal to $b_1, b_0, -b'_0, b_2$, respectively.

Let Φ be the submodule of $\Lambda_0 \oplus \Lambda'_0$ generated by the vectors $\{g + g'\}_{g \in G_0}$. Set $L = \Lambda_1 \oplus \Phi \oplus \Lambda_2 \subset \Lambda$. Observe that the projection $p : L \to \Lambda_1 \oplus \Lambda_2$ along Φ transforms B into $b_1 \oplus b_2$. Indeed, for any $g_1, h_1 \in G_1, g, h \in G_0, g_2, h_2 \in G_2$,

$$B(g_1 + g + g' + g_2,\ h_1 + h + h' + h_2)$$
$$= b_1(g_1, h_1) + b_0(g, h) + (-b'_0)(g', h') + b_2(g_2, h_2) = b_1(g_1, h_1) + b_2(g_2, h_2).$$

Pick a filling $\{\lambda_i\}_i \subset \Lambda_1 \oplus \Lambda_0$ of (T_1, T_0) whose matrix has rank $2\sigma(T_1, T_0)$. This means that the restriction of B to the submodule $V_1 \subset \Lambda_1 \oplus \Lambda_0$ generated by $\{\lambda_i\}_i$ has rank $2\sigma(T_1, T_0)$. Similarly, pick a filling $\{\varphi_j\}_j \subset \Lambda'_0 \oplus \Lambda_2$ of $(-T'_0, T_2)$ such that the restriction of B to the submodule $V_2 \subset \Lambda'_0 \oplus \Lambda_2$ generated by $\{\varphi_j\}_j$ has rank $2\sigma(-T'_0, T_2)$. We claim that there is a finite set $\psi \subset (V_1 + V_2) \cap L$ such that $p(\psi) \subset \Lambda_1 \oplus \Lambda_2$ is a filling of (T_1, T_2). Denoting by V the submodule of $\Lambda_1 \oplus \Lambda_2$ generated by $p(\psi)$, we obtain then the desired inequality:

$$\sigma(T_1, T_2) \leq \sigma(p(\psi)) = (1/2) \operatorname{rank}((b_1 \oplus b_2)|_V) = (1/2) \operatorname{rank}(B|_{p^{-1}(V)}) \leq (1/2) \operatorname{rank}(B|_{(V_1+V_2)\cap L})$$
$$\leq (1/2) \operatorname{rank}(B|_{V_1+V_2}) = (1/2) \operatorname{rank}(B|_{V_1}) + (1/2) \operatorname{rank}(B|_{V_2}) = \sigma(T_1, T_0) + \sigma(-T_0, T_2).$$

Here the second inequality follows from the inclusion $p^{-1}(V) \subset (V_1 + V_2) \cap L + \operatorname{Ker} p \subset L$ and the fact that $\operatorname{Ker} p = \Phi$ lies in the annihilator of $B|_L$.

To construct ψ, we modify $\{\lambda_i\}_i$ as follows. Let λ_1 be the vector of this filling equal to $s_1 + s_0$. Adding appropriate multiples of λ_1 to other λ_i we can ensure that the basis vector $s_0 \in G_0$ appears in all $\{\lambda_i\}_{i \neq 1}$ with coefficient 0. This transforms $\{\lambda_i\}_i$ into a new filling of (T_1, T_0) which will be from now on denoted $\lambda = \{\lambda_i\}_i$. This transformation does not change the module V_1 generated by $\{\lambda_i\}_i$. Similarly, we can assume that a vector φ_1 of the filling $\varphi = \{\varphi_j\}_j$ is equal to $s'_0 + s_2$ and the basis vector $s'_0 \in G'_0$ appears in all $\{\varphi_j\}_{j \neq 1}$ with coefficient 0,

The filling λ gives rise to a 1-dimensional manifold Γ_λ with boundary $(G_1 \cup G_0) - \{s_1, s_0\}$. Each λ_i having the form $g + h (\operatorname{mod} Rs_1)$ with $g, h \in (G_1 \cup G_0) - \{s_1, s_0\}$ gives rise to a component of Γ_λ homeomorphic to

$[0,1]$ and connecting g with h. Each λ_i having the form $g(\bmod Rs_1)$ with $g \in (G_1 \cup G_0) - \{s_1, s_0\}$ gives rise to a component of Γ_λ which is a copy of $[0,\infty)$ where 0 is identified with g. Other λ_i and in particular λ_1 do not contribute to Γ_λ. The definition of a filling implies that $\partial\Gamma_\lambda = (G_1 \cup G_0) - \{s_1, s_0\}$. Similarly, the filling φ gives rise to a 1-dimensional manifold Γ_φ with boundary $(G'_0 \cup G_2) - \{s'_0, s_2\}$. We can assume that Γ_λ and Γ_φ are disjoint. Gluing Γ_λ to Γ_φ along the canonical identification $G_0 - \{s_0\} \to G'_0 - \{s'_0\}, g \mapsto g'$, we obtain a 1-dimensional manifold, Γ, with $\partial\Gamma = (G_1 - \{s_1\}) \cup (G_2 - \{s_2\})$. Each component K of Γ is glued from several components of $\Gamma_\lambda \amalg \Gamma_\varphi$ associated with certain vectors $\lambda_i \in V_1 \subset \Lambda_1 \oplus \Lambda_0 \subset \Lambda$ and/or $\varphi_j \in V_2 \subset \Lambda'_0 \oplus \Lambda_2 \subset \Lambda$. Let $\psi_K \in \Lambda$ be the sum of these vectors. Observe that $\psi_K \in (V_1 + V_2) \cap L$; the inclusion $\psi_K \in L$ follows from two facts: (i) each point of $K \cap (G_0 - \{s_0\}) \approx K \cap (G'_0 - \{s'_0\})$ is adjacent to one component of Γ_λ and to one component of Γ_φ and (ii) s_0 does not show up in $\{\lambda_i\}_{i\neq 1}$ and s'_0 does not show up in $\{\varphi_j\}_{j\neq 1}$. Set $\psi_1 = \lambda_1 + \varphi_1 = s_1 + s_0 + s'_0 + s_2 \in \Lambda$. Clearly, $\psi_1 \in (V_1 + V_2) \cap L$. Set $\psi = \{\psi_1\} \cup \{\psi_K\}_K$ where K runs over the components of Γ with non-void boundary. Let us check that $p(\psi) \subset \Lambda_1 \oplus \Lambda_2$ is a filling of (T_1, T_2). Observe that for a compact component K of Γ with endpoints $g, h \in G_1 \cup G_2$, we have $p(\psi_K) = g + h \,(\bmod Rs_1 + Rs_2)$. For a non-compact component K of Γ with one endpoint $g \in G_1 \cup G_2$, we have $p(\psi_K) = g(\bmod Rs_1 + Rs_2)$. Thus all vectors in the family ψ are short and their sum is equal to $\sum_{g \in G_1 \cup G_2} g(\bmod Rs_1 + Rs_2)$. Also $p(\psi_1) = s_1 + s_2$. This means that $p(\psi)$ is a filling of (T_1, T_2) so that ψ satisfies all the required conditions. □

7.3. **Cobordism of based matrices.** Two based matrices T_1, T_2 over R are *cobordant* if $\sigma(T_1, -T_2) = 0$.

Theorem 7.3.1. *(i) Cobordism is an equivalence relation on the set of isomorphism classes of based matrices.*

(ii) Homologous based matrices are cobordant.

(iii) The genus of a tuple of based matrices is a cobordism invariant.

(iv) A based matrix is cobordant to a trivial based matrix if and only if it is hyperbolic.

Proof. (i) For a based matrix $T = (G, s, b)$, the based matrix $-T$ is isomorphic to the triple (G', s', b') where $G' = \{g' \mid g \in G\}$ is a disjoint copy of G and $b'(g', h') = -b(g, h)$ for any $g, h \in G$. Consider the filling $\{g + g'\}_{g \in G}$ of the pair $(T, -T)$. The matrix of this filling is 0. Therefore $\sigma(T, -T) = 0$ so that T is cobordant to itself. The symmetry of cobordism follows from the equalities $\sigma(T_2, -T_1) = \sigma(-T_2, T_1) = \sigma(T_1, -T_2)$. The transitivity of cobordism follows from the inequalities

$$0 \leq \sigma(T_1, -T_3) \leq \sigma(T_1, -T_2) + \sigma(T_2, -T_3)$$

which is a special case of Lemma 7.2.1.

(ii) Let a based matrix T' be obtained from a based matrix $T = (G, s, b)$ by a move M_i with $i = 1, 2, 3$. We can assume that the underlying set of T' is a union of a disjoint copy $\{h' \mid h \in G\}$ of G and one new element g in the case $i = 1, 2$ or two new elements g_1, g_2 in the case $i = 3$. For $i = 1$ (resp. $i = 2$, 3), the vectors $\{h + h'\}_{h \in G}$ and the vector g (resp. $g - s'$, $g_1 + g_2 - s'$) form a filling of the pair $(T, -T')$. The matrix of this filling is zero. Hence $\sigma(T, -T') = 0$ so that T is cobordant to T'.

(iii) We need to prove that $\sigma(T_1, ..., T_r)$ is preserved when $T_1, ..., T_r$ are replaced with cobordant based matrices. By induction, it suffices to prove that $\sigma(T_1, ..., T_{r-1}, T'_r) = \sigma(T_1, ..., T_{r-1}, T_r)$ for any based matrix T'_r cobordant to T_r. Lemma 7.2.1 gives that

$$\sigma(T_1, ..., T_{r-1}, T_r) \leq \sigma(T_1, ..., T_{r-1}, T'_r) + \sigma(-T'_r, T_r) = \sigma(T_1, ..., T_{r-1}, T'_r).$$

Similarly, $\sigma(T_1, ..., T_{r-1}, T'_r) \leq \sigma(T_1, ..., T_{r-1}, T_r)$. Hence $\sigma(T_1, ..., T_{r-1}, T'_r) = \sigma(T_1, ..., T_{r-1}, T_r)$.

(iv) If a based matrix T is cobordant to a trivial based matrix $T_0 = (\{s_0\}, s_0, b = 0)$, then $\sigma(T) = \sigma(T_0) = 0$ and therefore T is hyperbolic. Conversely, if $T = (G, s, b)$ is hyperbolic, then it has a filling with zero matrix. Adding to this filling the vector $s + s_0$, we obtain a filling of the pair (T, T_0) with zero matrix. Hence T is cobordant to $-T_0 = T_0$. □

Corollary 7.3.2. *For any based matrices $T_1, ..., T_r$ over R, we have $\sigma((T_1)_\bullet, ..., (T_r)_\bullet) = \sigma(T_1, ..., T_r)$.*

7.4. **Exercises.** 1. Verify that the definitions of the genus of a (single) based matrix over R given in Sections 7.1 and 7.2 are equivalent.

2. Prove that the function $(T_1, T_2) \mapsto \sigma(T_1, -T_2)$ defines a metric on the set of cobordism classes of based matrices over R.

3. Prove that $\sigma(T_1^-, ..., T_r^-) = \sigma(T_1, ..., T_r)$ for any based matrices $T_1, ..., T_r$ over R.

4. Prove that for any $1 \leq t \leq r$ and any based matrices $T_1, ..., T_r$, $T'_1, ..., T'_q$ with $q \geq 1$,

$$\sigma(T_1, ..., T_r) \leq \sigma(T_1, ..., T_t, T'_1, ..., T'_q) + \sigma(-T'_1, ..., -T'_q, T_{t+1}, ..., T_r) + q - 1.$$

5. Prove that $u(T_1)+...+u(T_r)=0$ for any hyperbolic tuple of based matrices $T_1,...,T_r$ over $\mathbb{R}$.

8. Genus estimates and sliceness of strings

8.1. **Genus estimates for strings.** Setting $R=\mathbb{Z}$, we can apply the definitions and results of Section 7 to the based matrices of strings. We begin with an estimate relating the slice genus of strings to the genus of their based matrices.

Lemma 8.1.1. *For any string α, we have $\sigma(T_\bullet(\alpha))=\sigma(T(\alpha))\leq 2\,sg(\alpha)$.*

Proof. The equality $\sigma(T_\bullet(\alpha))=\sigma(T(\alpha))$ follows from Corollary 7.1.2. We prove that $\sigma(T(\alpha))\leq 2\,sg(\alpha)$.

Consider an oriented 3-manifold M, a compact oriented surface F of genus $k=sg(\alpha)$ bounded by a circle S, and a proper map $\omega: F\to M$ such that $\omega|_S: S\to\partial M$ is a (generic) closed curve on ∂M realizing α. Let $\mathrm{in}: H_1(\partial M)\to H_1(M)$ be the inclusion homomorphism and $\omega_*: H_1(F)\to H_1(M)$ be the homomorphism induced by ω. Set $L=\mathrm{in}^{-1}(\omega_*(H_1(F)))\subset H_1(\partial M)$. Since the intersection form $B: H_1(\partial M)\times H_1(\partial M)\to\mathbb{Z}$ annihilates the kernel of in and $\operatorname{rank}\omega_*(H_1(F))\leq\operatorname{rank}H_1(F)=2k$, we obtain that the rank of the bilinear form $B|_L: L\times L\to\mathbb{Z}$ is smaller than or equal to $4k$.

Consider the based matrix $T=T(\alpha)=(G,s,b)$ of α. As in the proof of Lemma 5.1.5, the map ω gives rise to an involution ν on the set $\bowtie(\omega(S))=\mathrm{arr}(\alpha)=G-\{s\}$. This defines a simple filling $\mathcal{X}$ of T consisting of $\{s\}$ and the orbits of ν. The proof of Lemma 5.1.5 shows that $\sum_{x\in X}[\omega_x]\in L$ for any orbit X of ν. The homology class $[\omega(S)]\in H_1(\partial M)$ also lies in L because $\mathrm{in}([\omega(S)])=0$. The matrix of $\mathcal{X}$ is obtained by evaluating B on the vectors $[\omega(S)]$ and $\{\sum_{x\in X}[\omega_x]\}_X$ where X runs over the orbits of ν. Therefore the rank of this matrix is smaller than or equal to $\operatorname{rank}(B|_L)\leq 4k$. Hence $\sigma(T)\leq 2k=2\,sg(\alpha)$. □

The following theorem provides an algebraic obstruction to the sliceness of a string.

Theorem 8.1.2. *For a slice string α, the based matrices $T(\alpha)$ and $T_\bullet(\alpha)$ are hyperbolic.*

This theorem is a direct consequence of the previous lemma and the definitions. We complement this theorem with the following result whose proof is postponed to Section 8.3.

Theorem 8.1.3. *Based matrices of cobordant strings are cobordant.*

8.2. **Genus estimates for sequences of strings.** We generalize Lemma 8.1.1 to sequences of strings.

Lemma 8.2.1. *For any strings $\alpha_1,...,\alpha_r$,*

$$\sigma(T_\bullet(\alpha_1),...,T_\bullet(\alpha_r))=\sigma(T(\alpha_1),...,T(\alpha_r))\leq 2\,sg(\alpha_1,...,\alpha_r).$$

Proof. The equality $\sigma(T_\bullet(\alpha_1),...,T_\bullet(\alpha_r))=\sigma(T(\alpha_1),...,T(\alpha_r))$ follows from Corollary 7.3.2. The rest of the proof is similar to the proof of Lemma 8.1.1. Consider an oriented 3-manifold M, a compact (oriented) surface F of genus $k=sg(\alpha_1,...,\alpha_r)$ bounded by r circles $S_1,...,S_r$, and a proper map $\omega: F\to M$ such that the maps $\omega|_{S_t}: S_t\to\partial M$ with $t=1,...,r$ are disjoint (generic) closed curves on ∂M realizing $\alpha_1,...,\alpha_r$, respectively. Let $\mathrm{in}: H_1(\partial M)\to H_1(M)$ be the inclusion homomorphism and $\omega_*: H_1(F)\to H_1(M)$ be the homomorphism induced by ω. The group $H_1(F)$ is generated by the homology classes of $S_1,...,S_r\subset F$ and a subgroup $H\subset H_1(F)$ isomorphic to $\mathbb{Z}^{2k}$. Set $L=\mathrm{in}^{-1}(\omega_*(H))\subset H_1(\partial M)$. Since the intersection form $B: H_1(\partial M)\times H_1(\partial M)\to\mathbb{Z}$ annihilates the kernel of in, we obtain that

$$\operatorname{rank}(B|_L: L\times L\to\mathbb{Z})\leq 2\operatorname{rank}\omega_*(H)\leq 2\operatorname{rank}H=4k.$$

For $t=1,...,r$, consider the based matrix $T_t=(G_t,s_t,b_t)$ of α_t. Set $G=\cup_t G_t$. As in the proof of Lemma 5.1.5, the map ω gives rise to an involution ν on the set $\bowtie(\omega(\partial F))=G-\{s_1,...,s_r\}$. The proof of Lemma 5.1.5 shows that for any orbit X of ν, we have $\sum_{x\in X}[\omega_x]\in\mathrm{in}^{-1}(\omega_*(H_1(F)))$. Adding to $\sum_{x\in X}[\omega_x]$ an appropriate linear combination $\sum_t n_{X,t}[\omega(S_t)]$ of the homology classes $[\omega(S_1)],...,[\omega(S_r)]\in H_1(\partial M)$ with $n_{X,t}\in\mathbb{Z}$ we obtain an element of L. Consider the vector $\sum_{x\in X}x+\sum_t n_{X,t}s_t$ in the lattice $\mathbb{Z}G$ freely generated by G. These vectors corresponding to all orbits X of ν together with the vector $s_1+...+s_r\in\mathbb{Z}G$ form a filling of the tuple $T_1,...,T_r$. The matrix of this filling is obtained by evaluating B on the homology classes $\{\sum_{x\in X}[\omega_x]+\sum_t n_{X,t}[\omega(S_t)]\in H_1(\partial M)\}_X$ and $[\omega(S_1)]+...+[\omega(S_r)]\in H_1(\partial M)$. Since all these homology classes belong to L, the rank of this matrix is smaller than or equal to $4k$. Thus $\sigma(T(\alpha_1),...,T(\alpha_r))\leq 2k=2\,sg(\alpha_1,...,\alpha_r)$. □

Theorem 8.2.2. *If a sequence of strings is slice, then the sequence of their based matrices and the sequence of their primitive based matrices are hyperbolic.*

This theorem is a direct consequence of the previous lemma and the definitions.

8.3. **Proof of Theorem 8.1.3.** If strings α, β are cobordant, then $sg(\alpha, \overline{\beta}^-) = 0$. By Lemma 8.2.1, $\sigma(T(\alpha), T(\overline{\beta}^-)) = 0$. As we know, $T(\overline{\beta}^-) = -T(\beta)$. Thus, $\sigma(T(\alpha), -T(\beta)) = 0$ so that $T(\alpha)$ is cobordant to $T(\beta)$.

8.4. **Secondary obstructions to sliceness.** We introduce invariants of strings which may give further obstructions to sliceness (cf. Question 2 in Section 13). Consider a string α with core circle S and canonical realization $\omega : S \to \Sigma_\alpha$ as in Section 4.1. Let Σ be the closed oriented surface obtained by gluing 2-disks to all components of $\partial\Sigma_\alpha$. Pick an integer $p \geq 2$ and set $R = \mathbb{Z}/p\mathbb{Z}$ and $H = H_1(\Sigma; R)$. The R-module H is generated by the set $s \cup \{[e]\}_{e \in \operatorname{arr}(\alpha)}$ where the homology classes of loops on Σ are taken with coefficients in R and $s = [\omega(S)] \in H$, cf. Section 4.2. Consider the intersection form $B_R : H \otimes H \to R$. For $h \in H$, consider the string α_h formed by S and the arrows $e \in \operatorname{arr}(\alpha)$ such that $B_R([e], h) = 0$. The invariants of α_h can be viewed as invariants of α parametrized by p and h. In particular, we can consider the 1-variable polynomial $u(\alpha_h)$, the based matrix $T(\alpha_h)$, etc.

In the next lemma, a *Lagrangian* is a group $L \subset H$ equal to its annihilator $\operatorname{Ann}(L) = \{g \in H \mid B_R(L, g) = 0\}$. If p is prime, then each Lagrangian $L \subset H$ is a direct summand of H and $H/L \approx L$.

Lemma 8.4.1. *If α is slice, then there is a Lagrangian $L \subset H$ such that $s \in L$ and the string α_h is slice for all $h \in L$. Moreover, there is an involution on the set* $\operatorname{arr}(\alpha)$ *such that for any its orbit X, $\sum_{e \in X}[e] \in L$.*

Proof. If α is slice, then there are a compact oriented 3-manifold M' and a realization $\omega' : S \to \partial M'$ of α contractible in M'. The pair $(\partial M', \omega')$ can be obtained from $\omega : S \to \Sigma$ of α by 1-surgeries on $\Sigma - \omega(S)$. Attaching the corresponding solid 1-handles to $\Sigma \times 0 \subset \Sigma \times [0, 1]$ we obtain an oriented 3-manifold N such that $\partial N = (-\partial M') \cup \Sigma$ and the curves ω', ω are homotopic in N. Gluing N to M' along $\partial M'$ we obtain a compact oriented 3-manifold M such that $\partial M = \Sigma$ and ω is contractible in M. Consider the boundary homomorphism $\partial : H_2(M, \partial M; R) \to H_1(\partial M; R) = H$ and the inclusion homomorphism $i : H = H_1(\partial M; R) \to H_1(M; R)$. Set $L = \operatorname{Im}(\partial) = \operatorname{Ker}(i)$. It is well known that L is a Lagrangian. For completeness, we outline a proof. An element $g \in H$ belongs to $\operatorname{Ann}(L)$ iff $B_R(\partial x, g) = 0$ for every $x \in H_2(M, \partial M; R)$. By the Poincaré duality, there is a unique $\tilde{x} \in H^1(M; R) = \operatorname{Hom}(H_1(M; R), R)$ such that $x = \tilde{x} \cap [M]$. Then $B_R(\partial x, g) = x \cdot i(g) = \tilde{x}(i(g))$ where $\cdot$ is the intersection pairing $H_2(M, \partial M; R) \times H_1(M; R) \to R$. Therefore $g \in \operatorname{Ann}(L)$ iff $i(g)$ is annihilated by all homomorphisms $H_1(M; R) \to R$. This holds iff $i(g) = 0$, that is iff $g \in L$.

Pick $h \in L$ and pick any x in $\partial^{-1}(h) \subset H_2(M, \partial M; R)$. The cohomology class $\tilde{x} \in H^1(M; R)$ defines a p-fold covering $\tilde{M} \to M$. Since $\omega : S \to \Sigma = \partial M$ is contractible in M, it lifts to a loop $\tilde{\omega} : S \to \partial\tilde{M}$ contractible in $\tilde{M}$. By the equality $\tilde{x}(i(g)) = B_R(h, g)$ for $g \in H$, the underlying string of $\tilde{\omega}$ is α_h. Therefore α_h is slice. Constructing an involution on $\operatorname{arr}(\alpha)$ as in the proof of Lemma 5.1.5 (where F is a 2-disk) we obtain the last claim of the lemma. $\square$

This lemma implies that for all $h \in L$, the based matrix $T(\alpha_h)$ is hyperbolic and $u(\alpha_h) = 0$.

9. Lie cobracket for strings

We introduce a Lie cobracket in the free module generated by homotopy classes of strings. This induces a Lie bracket in the module of homotopy invariants of strings and other related algebraic structures.

Throughout the section, we fix a commutative ring with unit R.

9.1. **Lie coalgebras.** We recall here the notion of a Lie coalgebra dual to the one of a Lie algebra. To this end, we first reformulate the notion of a Lie algebra. For an R-module L, denote by Perm_L the permutation $x \otimes y \mapsto y \otimes x$ in $L^{\otimes 2} = L \otimes L$ and by τ_L the permutation $x \otimes y \otimes z \mapsto z \otimes x \otimes y$ in $L^{\otimes 3} = L \otimes L \otimes L$. Here and below $\otimes = \otimes_R$. A Lie algebra over R is an R-module L endowed with an R-homomorphism (the Lie bracket) $\theta : L^{\otimes 2} \to L$ such that $\theta \circ \operatorname{Perm}_L = -\theta$ (antisymmetry) and

$$\theta \circ (\operatorname{id}_L \otimes \theta) \circ (\operatorname{id}_{L^{\otimes 3}} + \tau_L + \tau_L^2) = 0 \in \operatorname{Hom}_R(L^{\otimes 3}, L)$$

(the Jacobi identity). Dually, a Lie coalgebra over R is an R-module A endowed with an R-homomorphism (the Lie cobracket) $\nu : A \to A^{\otimes 2}$ such that $\operatorname{Perm}_A \circ \nu = -\nu$ and

$$(\operatorname{id}_{A^{\otimes 3}} + \tau_A + \tau_A^2) \circ (\operatorname{id}_A \otimes \nu) \circ \nu = 0 \in \operatorname{Hom}_R(A, A^{\otimes 3}). \tag{9.1.1}$$

For a Lie coalgebra $(A, \nu : A \to A^{\otimes 2})$ over R and an integer $n \geq 1$, set

$$\nu^{(n)} = (\operatorname{id}_A^{\otimes (n-1)} \otimes \nu) \circ \cdots \circ (\operatorname{id}_A \otimes \nu) \circ \nu : A \to A^{\otimes (n+1)}.$$

In particular, $\nu^{(1)} = \nu$. Following [Tu2], Section 11, we call a Lie coalgebra (A, ν) over R *spiral*, if A is free as an R-module and the filtration $\operatorname{Ker}\nu^{(1)} \subset \operatorname{Ker}\nu^{(2)} \subset \cdots$ exhausts A, i.e., $A = \cup_{n\geq 1}\operatorname{Ker}\nu^{(n)}$.

A Lie coalgebra (A, ν) gives rise to the *dual Lie algebra* $A^* = \operatorname{Hom}_R(A, R)$ where the Lie bracket $A^*\otimes A^* \to A^*$ is the homomorphism dual to ν. For $u, v \in A^*$, the value of $[u, v] \in A^*$ on $x \in A$ is computed by

$$[u, v](x) = \sum_i u(x_i^{(1)})\, v(x_i^{(2)}) \in R$$

for any (finite) expansion $\nu(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \in A \otimes A$.

A *homomorphism* of Lie coalgebras $(A, \nu) \to (A', \nu')$ is an R-linear homomorphism $\psi : A \to A'$ such that $(\psi \otimes \psi)\nu(a) = \nu'\psi(a)$ for all $a \in A$. It is clear that the dual homomorphism $\psi^* : (A')^* \to A^*$ is a homomorphism of Lie algebras.

9.2. **Lie coalgebra of strings.** Let $\mathcal{S}$ be the set of homotopy classes of virtual strings and let $\mathcal{S}_0 \subset \mathcal{S}$ be its subset formed by the homotopically non-trivial classes. Let $\mathcal{A}_0 = \mathcal{A}_0(R)$ be the free R-module freely generated by $\mathcal{S}_0$. We shall provide $\mathcal{A}_0$ with the structure of a Lie coalgebra.

We begin with notation. For a string α, let $\langle\alpha\rangle$ denote its class in $\mathcal{S}_0$ if α is homotopically non-trivial and set $\langle\alpha\rangle = 0 \in \mathcal{A}_0$ if α is homotopically trivial. For an arrow $e = (a, b)$ of a string α, denote by α_e^1 the string obtained from α by removing all arrows except those with both endpoints in the interior of the arc ab. (In particular, e is removed.) Similarly, denote by α_e^2 the string obtained from α by removing all arrows except those with both endpoints in the interior of ba. Set

$$\nu(\langle\alpha\rangle) = \sum_{e\in\operatorname{arr}(\alpha)} \langle\alpha_e^1\rangle \otimes \langle\alpha_e^2\rangle - \langle\alpha_e^2\rangle \otimes \langle\alpha_e^1\rangle \in \mathcal{A}_0 \otimes \mathcal{A}_0. \tag{9.2.1}$$

Lemma 9.2.1. *The R-linear homomorphism $\nu : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{A}_0$ given on the generators of $\mathcal{A}_0$ by Formula 9.2.1 is a well-defined Lie cobracket. The Lie coalgebra $(\mathcal{A}_0, \nu)$ is spiral.*

Proof. To show that ν is well-defined we must verify that $\nu(\langle\alpha\rangle)$ does not change under the homotopy moves $(a)_s$, $(b)_s$, $(c)_s$ on α. The arrow added by $(a)_s$ contributes 0 to the cobracket by the definition of $\langle ... \rangle$. The contribution of all the other arrows is preserved. Similarly, the two arrows added by $(b)_s$ contribute opposite terms to the cobracket which is therefore preserved. Under $(c)_s$, all arrows contribute the same before and after the move.

The equality $\operatorname{Perm}_{\mathcal{A}_0} \circ\nu = -\nu$ is obvious. We now verify Formula 9.1.1. Let α be a string with core circle S. We can expand $(\operatorname{id}\otimes\nu)(\nu(\langle\alpha\rangle))$ as a sum of expressions $z(e, f)$ associated with ordered pairs of unlinked arrows $e, f \in \operatorname{arr}(\alpha)$. Note that the endpoints of e, f split S into four arcs meeting only at their endpoints. The endpoints of e (resp. f) bound one of these arcs, say x (resp. y). The other two arcs form $S - (x \cup y)$ and lie "between" e and f. Denote by β (resp. γ, δ) the string obtained from α by removing all arrows except those with both endpoints in the interior of x (resp. of y, of $S - (x \cup y)$). Set $\varepsilon = +1$ if e and f are co-oriented, i.e., if their tails bound a component of $S - (x \cup y)$. It is easy to see that

$$z(e, f) = \varepsilon(\langle\beta\rangle \otimes \langle\delta\rangle \otimes \langle\gamma\rangle - \langle\beta\rangle \otimes \langle\gamma\rangle \otimes \langle\delta\rangle).$$

A direct computation using this formula gives

$$(\operatorname{id}_{\mathcal{A}_0^{\otimes 3}} + \tau_{\mathcal{A}_0} + \tau_{\mathcal{A}_0}^2)(z(e, f) + z(f, e)) = 0.$$

Thus $\operatorname{id}_{\mathcal{A}_0^{\otimes 3}} + \tau_{\mathcal{A}_0} + \tau_{\mathcal{A}_0}^2$ annihilates $(\operatorname{id}\otimes\nu)(\nu(\langle\alpha\rangle))$. Hence ν is a Lie cobracket. The spirality of $(\mathcal{A}_0, \nu)$ follows from the obvious fact that $\nu^{(n)}(\langle\alpha\rangle) = 0$ for any string α of rank $\leq n$. (Actually a stronger assertion holds: $\nu^{(n)}(\langle\alpha\rangle) = 0$ for any string α of rank $\leq 4n + 2$.) □

Let $\mathcal{A} = \mathcal{A}(R)$ be the free R-module freely generated by $\mathcal{S}$. Since $\mathcal{S} = \mathcal{S}_0 \cup \{O\}$ where $O \in \mathcal{S}$ is the homotopy class of a trivial string, $\mathcal{A} = \mathcal{A}_0 \oplus RO$. The Lie cobracket ν in $\mathcal{A}_0$ extends to $\mathcal{A}$ by $\nu(O) = 0$.

The Lie cobrackets in $\mathcal{A}_0$ and $\mathcal{A}$ induce Lie brackets in $\mathcal{A}_0^* = \operatorname{Hom}_R(\mathcal{A}_0, R)$ and $\mathcal{A}^* = \operatorname{Hom}_R(\mathcal{A}, R)$. Examples below show that these Lie cobrackets and Lie brackets are non-zero. Clearly, $\mathcal{A}^* = \mathcal{A}_0^* \oplus R$ where the Lie bracket in R is zero. The elements of $\mathcal{A}^*$ bijectively correspond to maps $\mathcal{S} \to R$, i.e., to R-valued homotopy invariants of strings. Thus, such invariants form a Lie algebra.

9.3. **Examples.** (1) If $\operatorname{rank}\alpha \leq 6$, then $\nu(\langle\alpha\rangle)=0$. This follows from the fact that any string of rank ≤ 2 is homotopically trivial.

(2) For any $p,q\geq 1$, we have $\nu(\langle\alpha_{p,q}\rangle)=0$.

(3) Consider the string α_σ of rank 7 where σ is the permutation $(123)(4)(576)$ of the set $\{1,2,\ldots,7\}$. It follows from the definitions that $\nu(\langle\alpha_\sigma\rangle)=\langle\alpha_{1,2}\rangle\otimes\langle\alpha_{2,1}\rangle-\langle\alpha_{2,1}\rangle\otimes\langle\alpha_{1,2}\rangle$. As we know, $\alpha_{1,2}$ and $\alpha_{2,1}$ are homotopically non-trivial strings representing distinct generators of $\mathcal{A}$. Hence $\nu(\langle\alpha_\sigma\rangle)\neq 0$. This example can be used to show that the product of strings is not commutative even up to homotopy: there are strings γ,δ such that a product of γ,δ is not homotopic to a product of δ,γ. Drawing a picture of α_σ, one observes that α_σ is a product of $\delta=\alpha_{2,1}$ with a string, γ, of rank 4 obtained from $\alpha_{1,2}$ by adding a "small" arrow. Since γ has a small arrow, it is easy to form a product of γ with δ also having a small arrow. The resulting string, β, is homotopic to a string of rank 6. Hence $\nu(\langle\beta\rangle)=0$. Therefore α_σ is not homotopic to β.

(4) In generalization of the previous example pick any integers $p,q,p',q'\geq 1$ such that $p+q\geq 3, p'+q'\geq 3$. Consider the string $\alpha=\alpha_\sigma$ of rank $m=p+q+p'+q'+1$ where σ is the permutation of the set $\{1,2,\ldots,m\}$ defined by

$$\sigma(i)=\begin{cases} i+q, \text{ if } 1\leq i\leq p\\ i-p, \text{ if } p<i\leq p+q\\ i, \text{ if } i=p+q+1\\ i+q', \text{ if } p+q+1<i\leq p+q+1+p'\\ i-p', \text{ if } p+q+1+p'<i\leq m.\end{cases}$$

It follows from the definitions that

$$\nu(\langle\alpha\rangle)=\langle\alpha_{p',q'}\rangle\otimes\langle\alpha_{p,q}\rangle-\langle\alpha_{p,q}\rangle\otimes\langle\alpha_{p',q'}\rangle.$$

Clearly, $\nu(\langle\alpha\rangle)\neq 0$ unless $p=p'$ and $q=q'$.

(5) Consider the numerical invariants $u_1,u_2,\ldots\in\mathcal{A}^*$ constructed in Section 3.1. For $p,p'\geq 1$, we compute the value of $[u_p,u_{p'}]\in\mathcal{A}^*$ on the string $\alpha=\alpha(p,p',q,q')$ defined in the previous example. Assume for concreteness that the numbers p,p',q,q' are pairwise distinct. Then

$$[u_p,u_{p'}](\alpha)=u_p(\alpha_{p',q'})\,u_{p'}(\alpha_{p,q})-u_p(\alpha_{p,q})\,u_{p'}(\alpha_{p',q'})=0-(-q)(-q')=-qq'.$$

Hence $[u_p,u_{p'}]\neq 0$ for $p\neq p'$.

9.4. **Filtration of $\mathcal{A}_0$.** Assigning to a string its homotopy rang and homotopy genus (see Section 2.5) we obtain two maps $hr,hg:\mathcal{S}_0\to\mathbb{Z}$. For $r,g\geq 0$, set

$$\mathcal{S}_{r,g}=\{\alpha\in\mathcal{S}_0\,|\,hg(\alpha)\leq r,\quad hg(\alpha)\leq g\}.$$

This set is finite since there is only a finite number of strings of rank $\leq r$. The set $\mathcal{S}_{r,g}$ generates a submodule of $\mathcal{A}_0$ denoted $\mathcal{A}_{r,g}$. This submodule is a free R-module of rank $\#(\mathcal{S}_{r,g})$. Clearly,

$$\nu(\mathcal{A}_{r,g})\subset\bigoplus_{p,q\geq 0,p+q<r}\mathcal{A}_{p,g}\otimes\mathcal{A}_{q,g}\subset\mathcal{A}_{r,g}\otimes\mathcal{A}_{r,g}. \tag{9.4.1}$$

Thus, each $\mathcal{A}_{r,g}$ a Lie coalgebra. The inclusions $\mathcal{A}_{r,g}\hookrightarrow\mathcal{A}_{r',g'}$ for $r\leq r', g\leq g'$ make the family $\{\mathcal{A}_{r,g}\}_{r,g}$ into a direct spectrum of Lie coalgebras. The equality $\mathcal{A}_0=\cup_{r,g}\mathcal{A}_{r,g}$ shows that $\mathcal{A}_0=\operatorname{inj}\lim\{\mathcal{A}_{r,g}\}$.

The Lie cobracket in $\mathcal{A}_{r,g}$ induces a Lie bracket in $\mathcal{A}^*_{r,g}=\operatorname{Hom}_R(\mathcal{A}_{r,g},R)$. Formula 9.4.1 implies that this Lie algebra is nilpotent. Restricting maps $\mathcal{S}_0\to R$ to $\mathcal{S}_{r,g}$ we obtain a Lie algebra homomorphism $\mathcal{A}^*_0\to\mathcal{A}^*_{r,g}$. It is clear that $\mathcal{A}^*_0=\operatorname{proj}\lim\{\mathcal{A}^*_{r,g}\}$.

9.5. **Relations with Lie coalgebras of curves.** Let Σ be a connected surface and $\hat\pi$ be the set of homotopy classes of closed curves on Σ. (It can be identified with the set of conjugacy classes in $\pi=\pi_1(\Sigma)$.) There is a map $\psi:\hat\pi\to\mathcal{S}$ sending each homotopy class of curves into the homotopy class of the underlying strings. Clearly, $\psi(\hat\pi)=\cup_r\mathcal{S}_{r,g}$ where $g=g(\Sigma)$ is the genus of Σ. Observe that the mapping class group of Σ acts on $\hat\pi$ in the obvious way and ψ factors through the projection of $\hat\pi$ to the set of orbits of this action.

Let $Z=Z(R)$ be the free R-module with basis $\hat\pi$. The map $\psi:\hat\pi\to\mathcal{S}$ induces an R-linear homomorphism $Z\to\mathcal{A}$ whose image is equal to $\cup_r\mathcal{A}_{r,g}$. Composing this homomorphism with the projection $\mathcal{A}=\mathcal{A}_0\oplus RO\to\mathcal{A}_0$ we obtain an R-linear homomorphism $\psi_0:Z\to\mathcal{A}_0$.

The author defined in [Tu2], Section 8 a structure of a spiral Lie coalgebra in Z. (In fact Z is a Lie bialgebra, but we do not need it.) A direct comparison of the definitions shows that the map $\psi_0:Z\to\mathcal{A}_0$ is a homomorphism of Lie coalgebras.

9.6. **Associated algebraic structures.** In this section we suppose that $R \supset \mathbb{Q}$. A spiral Lie coalgebra (A, ν) over R naturally gives rise to a group $\operatorname{Exp} A^*$ and a Hopf algebra $S(A)$ over R, see [Tu2], Section 11. For completeness, we recall here these constructions.

Observe first that the dual Lie algebra $A^* = \operatorname{Hom}_R(A, R)$ has the following completeness property. Consider the lower central series $A^* = A^{*(1)} \supset A^{*(2)} \supset \cdots$ of A^* where $A^{*(n+1)} = [A^{*(n)}, A^*]$ for $n \geq 1$. Let $a_1, a_2, \ldots \in A^*$ be an infinite sequence such that for any $n \geq 1$ all terms of the sequence starting from a certain place belong to $A^{*(n)}$. Clearly, if $x \in \operatorname{Ker} \nu^{(n)}$ and $a \in A^{*(n+1)}$, then $a(x) = 0$. Since $A = \cup_n \operatorname{Ker} \nu^{(n)}$, the sum $a(x) = a_1(x) + a_2(x) + \cdots$ contains only a finite number of non-zero terms for every $x \in A$. Therefore $a(x)$ is a well-defined element of R. The formula $x \mapsto a(x) : A \to R$ defines an element of A^* denoted $a_1 + a_2 + \cdots$ and called the (infinite) sum of $a_1, a_2, \ldots$. A similar argument shows that $\cap_n A^{*(n)} = 0$ and the natural Lie algebra homomorphism $A^* \to \operatorname{proj\,lim}_n (A^*/A^{*(n)})$ is an isomorphism.

For $a, b \in A^*$, consider the sum

$$\mu(a, b) = a + b + \frac{1}{2}[a, b] + \frac{1}{12}([a, [a, b]] + [b, [b, a]]) + \cdots \in A^*$$

where the right-hand side is the Campbell-Hausdorff series for $\log(e^a e^b)$, see [Se]. The resulting mapping $\mu : A^* \times A^* \to A^*$ is a group multiplication in A^*. Here $a^{-1} = -a$ and 0 is the group unit. The group (A^*, μ) is denoted $\operatorname{Exp} A^*$. Heuristically, this is the "Lie group" with Lie algebra A^*. The equality $A^* = \operatorname{proj\,lim}_n (A^*/A^{*(n)})$ implies that the group $\operatorname{Exp} A^*$ is pro-nilpotent.

Consider the symmetric (commutative and associative) algebra of A:

$$S = S(A) = \oplus_{n \geq 0} S^n(A).$$

Here $S^0(A) = R$, $S^1(A) = A$, and $S^n(A)$ is the n-th symmetric tensor power of A for $n \geq 2$. The unit $1 \in R = S^0(A)$ is the unit of S. The group multiplication $\mu : A^* \times A^* \to A^*$ induces a comultiplication $S \to S \otimes S$ as follows. Since A is a free R-module, the natural map $A \to (A^*)^*$ extends to an embedding of S into the algebra of R-valued functions on A^*. We can identify S with the image of this embedding. Similarly, we can identify $S \otimes S$ with an algebra of R-valued functions on $A^* \times A^*$. It is easy to observe that for any $x \in S$, we have $x \circ \mu \in S \otimes S$. Indeed, it suffices to prove this for $x \in A$. Then $x \in \operatorname{Ker} \nu^{(n)}$ for some n so that x annihilates all but finite number of terms of the Campbell-Hausdorff series. Our claim follows then from the duality between the Lie bracket in A^* and the Lie cobracket ν. For example, if $n = 3$ and $\nu^{(2)}(x) = \sum_i \alpha_i \otimes \beta_i \otimes \gamma_i \in A^{\otimes 3}$, then

$$x \circ \mu = x \otimes 1 + 1 \otimes x + \frac{1}{2}\nu(x) + \frac{1}{12} \sum_i (\alpha_i \beta_i \otimes \gamma_i + \gamma_i \otimes \alpha_i \beta_i).$$

The formula $\Delta(x) = x \circ \mu$ defines a coassociative comultiplication in S. It has a counit $S \to R$ defined as the projection to $S^0(A) = R$. The antipode $S \to S$ is the algebra homomorphism sending any $x \in A$ into $-x \in A$. A routine check shows that S is a (commutative) Hopf algebra. Heuristically, it should be viewed as the Hopf algebra of R-valued functions on the group $\operatorname{Exp} A^*$ or as the Hopf dual of the universal enveloping algebra of A^*.

The construction of $\operatorname{Exp} A^*$ and $S(A)$ can be generalized as follows. Pick $h \in R$ and observe that the mapping $h\nu : A \to A \otimes A$ is a Lie cobracket in A. It induces the Lie bracket $[,]_h = h[,]$ in A^* where $[,]$ is the Lie bracket induced by ν. The corresponding multiplication μ_h in A^* is given by

$$\mu_h(a, b) = a + b + \frac{h}{2}[a, b] + \frac{h^2}{12}([a, [a, b]] + [b, [b, a]]) + \cdots$$

This multiplication makes A^* into a group denoted $\operatorname{Exp}_h A^*$. As above, μ_h induces a comultiplication in the symmetric algebra $S = S(A)$. This makes S into a Hopf algebra over R denoted $S_h(A)$. For $h = 1$, we obtain the same objects as in the previous paragraphs. Note for the record that for any $h \in R$, the formula $a \mapsto ha : A^* \to A^*$ defines a group homomorphism $\operatorname{Exp}_h A^* \to \operatorname{Exp} A^*$. If $h \in R$ is a non-zero-divisor, this homomorphism is injective.

We can apply the constructions of this subsection to any $h \in R$ and to the spiral Lie coalgebras $\mathcal{A}_0, \mathcal{A}, \mathcal{A}_{r,g}$, Z considered above. The equality $\mathcal{A} = \mathcal{A}_0 \oplus R$ implies that $\operatorname{Exp}_h \mathcal{A}^* = \operatorname{Exp}_h \mathcal{A}_0^* \times \underline{R}$ where $\underline{R}$ is the additive group of R. The group $\operatorname{Exp}_h \mathcal{A}_{r,g}^*$ and the Hopf algebra $S_h(\mathcal{A}_{r,g})$ are quotients of $\operatorname{Exp}_h \mathcal{A}_0^*$ and $S_h(\mathcal{A}_0)$, respectively. The homomorphism $\psi_0 : Z \to \mathcal{A}_0$ extends by multiplicativity to a Hopf algebra homomorphism $S_h(Z) \to S_h(\mathcal{A}_0)$. Dualizing ψ_0, we obtain a mapping $\mathcal{A}_0^* \to Z^*$ which is a Lie algebra homomorphism and at the same time a group homomorphism $\operatorname{Exp}_h \mathcal{A}_0^* \to \operatorname{Exp}_h Z^*$.

10. Virtual strings versus virtual knots

Virtual knots were introduced by L. Kauffman [Ka] as a generalization of classical knots. We relate them to virtual strings by showing that each virtual knot gives rise to a polynomial on virtual strings with coefficients in the ring $\mathbb{Q}[z]$. As a technical tool, we introduce a skein algebra of virtual knots and compute it in terms of strings.

10.1. **Virtual knots.** We define virtual knots in terms of arrow diagrams following [GPV]. An *arrow diagram* is a virtual string whose arrows are endowed with signs $\pm$. By the core circle and the endpoints of an arrow diagram, we mean the core circle and the endpoints of the underlying virtual string. The sign of an arrow e of an arrow diagram is denoted $\mathrm{sign}(e)$. Homeomorphisms of arrow diagrams are defined as the homeomorphisms of the underlying strings preserving the signs of all arrows. The homeomorphism classes of arrow diagrams will be also called arrow diagrams.

We describe three moves $(\mathrm{a})_{ad}$, $(\mathrm{b})_{ad}$, $(\mathrm{c})_{ad}$ on arrow diagrams where ad stands for "arrow diagram". Let α be an arrow diagram with core circle S. Pick two distinct points $a, b \in S$ such that the (positively oriented) arc $ab \subset S$ is disjoint from the set of endpoints of α. The move $(\mathrm{a})_{ad}$ adds to α the arrow (a, b) with sign $+$ or $-$. This move has two forms determined by the sign $\pm$. The move $(\mathrm{b})_{ad}$ acts on α as follows. Pick two arcs on S disjoint from each other and from the endpoints of α. Let a, a' be the endpoints of the first arc (in an arbitrary order) and b, b' be the endpoints of the second arc. The move adds to α two arrows (a, b) and (b', a') with opposite signs. This move has eight forms depending on the choice of the sign of (a, b), two possible choices for a, and two possible choices for b. (This list of eight forms of $(\mathrm{b})_{ad}$ contains two equivalent pairs so that in fact the move $(\mathrm{b})_{ad}$ has only six forms.) The move $(\mathrm{c})_{ad}$ applies to α when α has three arrows with signs $((a^+, b), +), ((b^+, c), +), ((c^+, a), -)$ where $a, a^+, b, b^+, c, c^+ \in S$ such that the arcs aa^+, bb^+, cc^+ are disjoint from each other and from the other endpoints of α. The move $(\mathrm{c})_{ad}$ replaces these three arrows with the arrows $((a, b^+), +), ((b, c^+), +), ((c, a^+), -)$.

By definition, a *virtual knot* is an equivalence class of arrow diagrams with respect to the equivalence relation generated by the moves $(\mathrm{a})_{ad}$, $(\mathrm{b})_{ad}$, $(\mathrm{c})_{ad}$ and homeomorphisms. Note that our set of moves is somewhat different from the one in [GPV] but generates the same equivalence relation (cf. below).

In the sequel the virtual knot represented by an arrow diagram D will be denoted $[D]$. A *trivial arrow diagram* having no arrows represents the *trivial virtual knot*.

Forgetting the signs of arrows, we can associate with any arrow diagram D its underlying virtual string $\underline{D}$. This induces a "forgetting" map $K \mapsto \underline{K}$ from the set of virtual knots into the set of virtual strings. This map is surjective bur not injective. The theory of virtual knots is considerably reacher than the theory of virtual strings. For instance, the fundamental group of a virtual knot [Ka] allows to distinguish virtual knots with the same underlying strings.

Note finally that the definition of an r-th covering of a string in Section 3.6 extends to virtual knots: one keeps only arrows e of an arrow diagram such that $n(e) \in r\mathbb{Z}$ and of course one keeps their signs.

10.2. **From knots to virtual knots.** Arrow diagrams are closely related to the standard knot diagrams on surfaces. An (oriented) knot diagram on an (oriented) surface Σ is a (generic oriented) closed curve on Σ such that at each its double point one of the branches of the curve passing through this point is distinguished. The distinguished branch is called an *overcrossing* while the second branch passing through the same point is called an *undercrossing*. A knot diagram on $\Sigma = \Sigma \times \{0\}$ determines an (oriented) knot in the cylinder $\Sigma \times \mathbb{R}$ by pushing the overcrossings into $\Sigma \times (0, \infty)$.

Any knot diagram d gives rise to an arrow diagram $D(d)$ as follows. First of all, the closed curve underlying d gives rise to a virtual string, see Section 2.2. We provide each arrow of this string with the sign of the corresponding double point of d. This sign is $+$ (resp. $-$) if the pair (a positive tangent vector to the overcrossing branch, a positive tangent vector to the undercrossing branch) is positive (resp. negative) with respect to the orientation of Σ. Our definition of the arrow diagram associated with d differs from the one in [GPV]: their arrow diagram is obtained from ours by reversing all arrows with sign $-$.

There is a canonical mapping from the set of isotopy classes of (oriented) knots in $\Sigma \times \mathbb{R}$ into the set of virtual knots. It assigns to a knot $K \subset \Sigma \times \mathbb{R}$ the virtual knot $[D(d)]$ where d is a knot diagram on Σ presenting a knot in $\Sigma \times \mathbb{R}$ isotopic to K. The virtual knot $[D(d)]$ does not depend on the choice of d. This follows from the fact that two knot diagrams on Σ presenting isotopic knots in $\Sigma \times \mathbb{R}$ can be obtained from each other by ambient isotopy in Σ and the Reidemeister moves. Recall the standard list of the Reidemeister moves: (1) a move adding a twist on the right (resp. left) of a branch; (2) a move pushing a branch over another branch

and creating two crossings; (3) a move pushing a branch over a crossing. This list is redundant. In particular, the left move of type (1) can be presented as a composition of type (2) moves and the inverse to a right move of type (1). One move of type (3) together with moves of type (2) is sufficient to generate all moves of type (3) corresponding to various orientations on the branches (see, for instance, [Tu1], pp. 543–544). As the generating move of type (3) we take the move $(c)^-$ described in Section 2.3. It remains to observe that the moves $(a)_{ad}$, $(b)_{ad}$, $(c)_{ad}$ on arrow diagrams are exactly the moves induced by the right Reidemeister moves of type (1), the Reidemeister moves of type (2), and the move $(c)^-$.

10.3. **Skein algebra of virtual knots.** Let $R = \mathbb{Q}[z]$ be the ring of polynomials in one variable z with rational coefficients. Consider the polynomial algebra $R[\mathcal{K}]$ generated by the set of virtual knots $\mathcal{K}$. This is a commutative associative algebra with unit whose elements are polynomials in elements of $\mathcal{K}$ with coefficients in R. We now introduce certain elements of $R[\mathcal{K}]$ called skein relations.

Pick an arrow diagram D with core circle S and pick an arrow $e = (a, b)$ of D with sign $+$ (here $a, b \in S$). Let D_e^- be the same arrow diagram with the sign of e changed to $-$. Let D'_e be the arrow diagram obtained from D by removing all arrows with at least one endpoint on the arc $ba \subset S$. Let D''_e be the arrow diagram obtained from D by removing all arrows with at least one endpoint on the arc $ab \subset S$. The *skein relation* corresponding to (D, e) is $[D] - [D_e^-] - z[D'_e][D''_e] \in R[\mathcal{K}]$.

The ideal of the algebra $R[\mathcal{K}]$ generated by the trivial virtual knot and the skein relations (determined by all the pairs (D, e) as above) is called the *skein ideal*. The quotient of $R[\mathcal{K}]$ by this ideal is called the *skein algebra of virtual knots* and denoted $\mathcal{E}$. The next theorem computes $\mathcal{E}$ in terms of strings. Recall the set $\mathcal{S}_0$ of non-trivial homotopy classes of virtual strings, cf. Section 9.2.

Theorem 10.3.1. *There is a canonical R-algebra isomorphism $\nabla : \mathcal{E} \to R[\mathcal{S}_0]$ where $R[\mathcal{S}_0]$ is the polynomial algebra generated by $\mathcal{S}_0$.*

This theorem allows us to associate with any virtual knot K a polynomial $\nabla(K) \in R[\mathcal{S}_0]$. It will be clear from the definitions that

$$\nabla(K) = \langle \underline{K} \rangle + \sum_{n \geq 2} z^{n-1} \nabla_n(K)$$

where $\nabla_n(K)$ is a homogeneous element of $\mathbb{Q}[\mathcal{S}_0]$ of degree n which is non-zero only for a finite set of n. Combining ∇ with homotopy invariants of strings we obtain invariants of virtual knots. For example, composing ∇ with the algebra homomorphism $R[\mathcal{S}_0] \to R[t]$ sending the homotopy class of a string α into the polynomial $u(\alpha)(t)$, we obtain an algebra homomorphism $\mathcal{E} \to R[t] = \mathbb{Q}[z, t]$. This gives a 2-variable polynomial invariant of virtual knots. Further polynomial invariants of virtual knots can be similarly obtained from the higher u-polynomials defined in Section 3.6.

The constructions above can be applied to the virtual knot derived from a geometric knot $K \subset \Sigma \times \mathbb{R}$ in Section 10.2. The resulting polynomial $\nabla(K) \in R[\mathcal{S}_0]$ is invariant under the action on knots of orientation preserving homeomorphisms $\Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ induced by orientation preserving homeomorphisms $\Sigma \to \Sigma$. The polynomial $\nabla(K)$ is interesting only in the case when the genus of Σ is at least 2. This is due to the fact that the strings realized by curves on a surface of genus 0 or 1 are homotopically trivial.

Theorem 10.3.1 will be proven in the next section. Here we give an explicit expression for the value of ∇ on the generator $[D] \in \mathcal{E}$ represented by an arrow diagram D. We need a few definitions. The endpoints of the arrows of D split the core circle of D into (oriented) arcs called the *edges* of D. Denote the set of edges of D by $\mathrm{edg}(D)$. Each endpoint a of an arrow of D is adjacent to two edges $a_-, a_+ \in \mathrm{edg}(D)$, respectively incoming and outgoing with respect to a. For an integer $n \geq 1$, an *n-labeling* of D is a map $f : \mathrm{edg}(D) \to \{1, 2, ..., n\}$ satisfying the following condition: for any arrow $e = (a, b)$ of D, either

(i) $f(a_+) = f(a_-), f(b_+) = f(b_-)$ or

(ii) $f(a_+) = f(b_-) \neq f(a_-) = f(b_+)$ and $\mathrm{sign}(f(a_-) - f(a_+)) = \mathrm{sign}(e)$.

The arrows e as in (ii) are called *f-cutting arrows*. The number of f-cutting arrows of D is denoted $|f|$ and the number of f-cutting arrows of D with sign $= -1$ is denoted $|f|_-$. Note that the value of f on two adjacent edges $a_-, a_+ \in \mathrm{edg}(D)$ may differ only when a is an endpoint of an f-cutting arrow. Therefore $|f| \geq \#f(\mathrm{edg}(D)) - 1$. For $i = 1, ..., n$, let $\underline{D}_{f,i}$ be the string obtained from D by removing all arrows except the arrows (a, b) with $f(a_+) = f(a_-) = f(b_+) = f(b_-) = i$ (and forgetting the signs of the arrows).

Let $\mathrm{lbl}_n(D)$ be the set of n-labelings f of D such that $f(\mathrm{edg}(D)) = \{1, ..., n\}$, $|f| = n-1$, and the f-cutting arrows of D are pairwise unlinked (in the sense of Section 3.1). Then

$$\nabla([D]) = \sum_{n=1} \sum_{f \in \mathrm{lbl}_n(D)} \frac{(-1)^{|f|_-} z^{n-1}}{n!} \prod_{i=1}^{n} \langle \underline{D}_{f,i} \rangle \in R[\mathcal{S}_0]. \tag{10.3.1}$$

The expression on the right-hand side is finite since $\mathrm{lbl}_n(D) = \emptyset$ for $n > \#\,\mathrm{edg}(D)$. The set $\mathrm{lbl}_1(D)$ consists of only one element $f = 1$ so that the free term of $\nabla([D])$ is $\langle \underline{D} \rangle$.

11. Proof of Theorem 10.3.1

The proof of Theorem 10.3.1 largely follows the proof of Theorems 9.2 and 13.2 in [Tu2]. We therefore expose only the main lines of the proof. The key point behind Theorem 10.3.1 is the existence of a natural comultiplication in $\mathcal{E}$ and we define it first. Then we construct ∇ and prove that it is an isomorphism.

11.1. **Comultiplication in $\mathcal{E}$.** We need to study more extensively the labelings of arrow diagrams defined at the end of the previous section. Let D be an arrow diagram with core circle S. Each n-labeling f of D gives rise to n monomials $D_{f,1}, ..., D_{f,n} \in \mathcal{E}$ as follows. Identifying $a = b$ for every f-cutting arrow (a, b) of D, we transform S into a 4-valent graph, Γ^f, with $|f|$ vertices. The projection $S \to \Gamma^f$ maps the non-f-cutting arrows of D into "arrows" on Γ^f, i.e., into ordered pairs of (distinct) generic points of Γ^f. The labeling f induces a labeling of the edges of Γ^f by the numbers $1, 2, ..., n$. It follows from the definition of a labeling that for each $i = 1, ..., n$, the union of edges of Γ^f labeled with i is a disjoint union of $r_i = r_i(f) \geq 0$ circles $S^i_1, ..., S^i_{r_i}$. The orientation of S induces an orientation of the edges of Γ_f and of these circles. We transform each circle S^i_j with $j = 1, ..., r_i$ into an arrow diagram by adding to it all the arrows of Γ^f with both endpoints on S^i_j. The signs of these arrows are by definition the signs of the corresponding non-f-cutting arrows of D. Set

$$D_{f,i} = \prod_{j=1}^{r_i} [S^i_j] \in \mathcal{E}.$$

For any $n \geq 2$, denote $\mathrm{Lbl}_n(D)$ the set of n-labelings f of D such that the f-cutting arrows of D are pairwise unlinked. The latter condition can be reformulated in terms of the numbers $r_1(f), ..., r_n(f)$ introduced above: $f \in \mathrm{Lbl}_n(D)$ if and only if $r_1(f) + \cdots + r_n(f) = |f| + 1$. For $f \in \mathrm{Lbl}_n(D)$, set

$$\Delta(D, f) = (-1)^{|f|_-} z^{|f|}\, D_{f,1} \otimes D_{f,2} \otimes \cdots \otimes D_{f,n} \in \mathcal{E}^{\otimes n}$$

where $\mathcal{E}^{\otimes n}$ is the tensor product over R of n copies of $\mathcal{E}$.

By a comultiplication in $\mathcal{E}$, we mean a coassociative algebra homomorphism $\Delta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$. (The coassociativity means that $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$.) We claim that the formula

$$\Delta([D]) = \sum_{f \in \mathrm{Lbl}_2(D)} \Delta(D, f) \in \mathcal{E} \otimes \mathcal{E}$$

extends by multiplicativity to a well-defined comultiplication in $\mathcal{E}$. This can be deduced from [Tu2], Theorem 9.2 or proven directly repeating the same arguments. We explain how to deduce our claim from [Tu2]. Comparing the definition of $\Delta([D])$ with the comultiplication in the algebra of skein classes of knots in (a surface)$\times\mathbb{R}$ given in [Tu2], we observe that they correspond to each other provided D underlies a knot diagram on the surface. (The variables $h = h_1, \overline{h} = h_{-1}$ used in [Tu2] should be replaced with 0 and z, respectively. After the substitution $h = 0$, we can consider only labelings satifying - in the notation of [Tu2] - the condition $||f|| = -|f|$ which translates here as the assumption that the f-cutting arrows of D are pairwise unlinked.) The results of [Tu2] imply that if a move $(\mathrm{a})_{ad}$, $(\mathrm{b})_{ad}$, $(\mathrm{c})_{ad}$ on D underlies a Reidemeister move on a knot diagram, then $\Delta([D])$ is preserved under this move. Since any arrow diagram D underlies a knot diagram on a surface and any move $(\mathrm{a})_{ad}$, $(\mathrm{b})_{ad}$, $(\mathrm{c})_{ad}$ on D can be induced by a Reidemeister move, we conclude that $\Delta([D])$ is invariant under the moves $(\mathrm{a})_{ad}$, $(\mathrm{b})_{ad}$, $(\mathrm{c})_{ad}$ on D. Therefore the formula $[D] \mapsto \Delta([D])$ yields a well-defined mapping $\mathcal{K} \to \mathcal{E} \otimes \mathcal{E}$. This mapping uniquely extends to an algebra homomorphism $R[\mathcal{K}] \to \mathcal{E} \otimes \mathcal{E}$. The results of [Tu2] imply that for an arrow diagram D underlying a knot diagram on a surface and any arrow e of D with $\mathrm{sign}(e) = +$, the skein relation $[D] - [D^-_e] - z[D'_e]\,[D''_e]$ lies in the kernel of the latter homomorphism. The condition that D underlies a knot diagram is verified for all D. Therefore the

homomorphism $R[\mathcal{K}] \to \mathcal{E}\otimes\mathcal{E}$ annihilates the skein ideal and induces an algebra homomorphism $\Delta : \mathcal{E} \to \mathcal{E}\otimes\mathcal{E}$. The coassociativity of Δ follows from the easy formulas

$$(\mathrm{id}\otimes\Delta)\Delta([D]) = \sum_{f\in \mathrm{Lbl}_3(D)} \Delta(D,f) = (\Delta\otimes \mathrm{id})\Delta([D])$$

(cf. [Tu2], p. 665). More generally, for any $n \geq 2$, the value on $[D] \in \mathcal{E}$ of the iterated homomorphism

$$\Delta^{(n)} = (\mathrm{id}^{\otimes(n-1)}\otimes\Delta)\circ(\mathrm{id}^{\otimes(n-2)}\otimes\Delta)\circ ...\circ(\mathrm{id}\otimes\Delta)\Delta : \mathcal{E} \to \mathcal{E}^{\otimes(n+1)}$$

is computed by

$$\Delta^{(n)}([D]) = \sum_{f\in \mathrm{Lbl}_{n+1}(D)} \Delta(D,f).$$

Note for the record that each arrow diagram D admits constant 2-labelings f_1, f_2 taking values $1, 2$ on all edges, respectively. The corresponding summands of $\Delta([D])$ are $\Delta(D,f_1) = [D]\otimes 1$ and $\Delta(D,f_2) = 1\otimes[D]$.

11.2. **Homomorphism $\nabla : \mathcal{E} \to R[\mathcal{S}_0]$.** There are two obvious R-linear homomorphisms $\varepsilon : \mathcal{E} \to R$ and $q : \mathcal{E} \to R[\mathcal{S}_0]$. The homomorphism ε sends $1 \in \mathcal{E}$ into $1 \in R$ and sends all virtual knots and their non-void products into 0. The homomorphism q sends 1 and all products of ≥ 2 virtual knots into 0 and sends a virtual knot K into $\langle \underline{K}\rangle$. Tensorizing q with itself, we obtain for all $n \geq 1$ a homomorphism $q^{\otimes n} : \mathcal{E}^{\otimes n} \to R[\mathcal{S}_0]^{\otimes n}$. Let $s_n : R[\mathcal{S}_0]^{\otimes n} \to R[\mathcal{S}_0]$ be the R-linear homomorphism sending $a_1\otimes\cdots\otimes a_n$ into $(n!)^{-1}a_1\cdots a_n$. Set

$$\nabla = \varepsilon + q + \sum_{n\geq 2} s_n\, q^{\otimes n}\Delta^{(n-1)} : \mathcal{E} \to R[\mathcal{S}_0]$$

where $\Delta^{(1)} = \Delta$. It is clear that ∇ is R-linear. The same argument as in [Tu2], Lemma 13.4 shows that ∇ is an algebra homomorphism. Computing ∇ on the skein class of an arrow diagram D, we obtain

$$\nabla([D]) = \sum_{n\geq 1} s_n\, q^{\otimes n} \sum_{f\in \mathrm{Lbl}_n(D)} \Delta(D,f) = \sum_{n\geq 1}\sum_{f\in \mathrm{Lbl}_n(D)} \frac{(-1)^{|f|_-}z^{|f|}}{n!}\prod_{i=1}^{n} q(D_{f,i}).$$

Note that $q(D_{f,i}) = 0$ unless $r_i(f) = 1$ in which case $q(D_{f,i}) = \langle \underline{D}_{f,i}\rangle$. For a labeling $f \in \mathrm{Lbl}_n(D)$ the equalities $r_1(f) = ... = r_n(f) = 1$ are equivalent to the inclusion $f \in \mathrm{lbl}_n(D)$. This yields Formula 10.3.1.

Observe that $\nabla([D])$ is a sum of $\langle \underline{D}\rangle$ and a polynomial in strings of rank $< \operatorname{rank} D$. An induction on the rank of strings shows that the image of ∇ contains all strings. Therefore ∇ is surjective.

The proof of the injectivity of ∇ is based on the following lemma.

Lemma 11.2.1. *There is a $\mathbb{Q}$-valued function η on the set of isomorphism classes of (finite) oriented trees such that the following holds:*

(i) if T is a tree with one vertex and no edges, then $\eta(T) = 1$;

(ii) if an oriented tree T' (resp. U) is obtained from an oriented tree T by reversing the orientation of an edge e (resp. by contracting e into a point), then $\eta(T) + \eta(T') + \eta(U) = 0$;

(iii) if an oriented tree T' (resp. T'') is obtained from an oriented tree T by replacing two distinct edges with common origin ab, ac by ab, bc (resp. by ac, cb) and if U is obtained from T by identifying b with c and ab with ac, then $\eta(T) = \eta(T') + \eta(T'') + \eta(U)$.

In this lemma by an edge ab we mean an *oriented* edge directed from a to b.

Lemma 11.2.1 was first established in [Tu2], Theorem 14.1 where it is also shown that η is unique (we shall not need this). The construction in [Tu2] is indirect and does not provide an explicit formula for η. Such a formula was pointed out by François Jaeger [Ja]. The following proof of Lemma 11.2.1 is a simplified version of the proof given by Jaeger [Ja].

Proof. By a *forest* we shall mean a disjoint union of a finite family of finite oriented trees. The set of vertices of a forest F is denoted $V(F)$. For a forest F and an integer $n \geq 1$, denote by $C_n(F)$ the set of surjective mappings $f : V(F) \to \{1, ..., n\}$ such that for every edge ab of F we have $f(a) < f(b)$. This set is empty for $n > \#(V(F))$. Set

$$\eta(F) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}\, \#(C_n(F)) \in \mathbb{Q}.$$

We claim that η satisfies all the conditions of the lemma. Condition (i) is obvious. Condition (iii) is a direct corollary of the definitions. Indeed for all n, the set $C_n(T)$ splits as a disjoint union of the sets

$C_n(T'), C_n(T''), C_n(U)$. Hence $\#(C_n(T)) = \#(C_n(T')) + \#(C_n(T'')) + \#(C_n(U))$ and $\eta(T) = \eta(T') + \eta(T'') + \eta(U)$. It remains to verify (ii). Let F be obtained from T by removing the interior of the edge e. For all n, the set $C_n(F)$ splits as a disjoint union of the sets $C_n(T), C_n(T'), C_n(U)$. Hence $\#(C_n(F)) = \#(C_n(T)) + \#(C_n(T')) + \#(C_n(U))$ and $\eta(F) = \eta(T) + \eta(T') + \eta(U)$. Thus we need only to prove that $\eta(F) = 0$ for every forest F with two components T_1, T_2.

For non-negative integers n, k_1, k_2, denote by $C_n(k_1, k_2)$ the set of pairs (l_1, l_2) where for $i = 1, 2$, l_i is an order-preserving injection from $\{1, ..., k_i\}$ into $\{1, ..., n\}$ and $l_1(\{1, ..., k_1\}) \cup l_2(\{1, ..., k_2\}) = \{1, ..., n\}$. Having $g_1 \in C_{k_1}(T_1)$, $g_2 \in C_{k_2}(T_2)$ and having $(l_1, l_2) \in C_n(k_1, k_2)$ we define a mapping $f = f(g_1, g_2, l_1, l_2) : V(F) \to \{1, ..., n\}$ by $f(v) = l_1 g_1(v)$ for $v \in V(T_1)$ and $f(v) = l_2 g_2(v)$ for $v \in V(T_2)$. Clearly, $f \in C_n(F)$. It is obvious that any $f \in C_n(F)$ can be uniquely presented in the form $f = f(g_1, g_2, l_1, l_2)$ where $g_i \in C_{k_i}(T_i)$ with $k_i = \#(f(V(T_i))) \geq 1$ for $i = 1, 2$. Therefore

$$\eta(F) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \left(\sum_{k_1, k_2 \geq 1} \sum_{g_1 \in C_{k_1}(T_1), g_2 \in C_{k_2}(T_2)} \#(C_n(k_1, k_2)) \right)$$

$$= \sum_{k_1, k_2 \geq 1} \#(C_{k_1}(T_1)) \, \#(C_{k_2}(T_2)) \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \#(C_n(k_1, k_2)).$$

Thus it is enough to prove that for all $k_1 \geq 1, k_2 \geq 1$, the numbers $c_n(k_1, k_2) = \#(C_n(k_1, k_2))$ verify

$$\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} c_n(k_1, k_2) = 0. \tag{11.2.1}$$

Clearly, $c_n(k_1, k_2)$ is the number of pairs (S_1, S_2) where S_1, S_2 are subsets of $\{1, ..., n\}$ such that $S_1 \cup S_2 = \{1, ..., n\}$, $\#(S_1) = k_1$, $\#(S_2) = k_2$. In particular, $c_n(k_1, k_2) = 0$ if $k_1 + k_2 < n$ or $k_1 > n$ or $k_2 > n$. For any $n \geq 1$ and commuting variables x, y,

$$(x + y + xy)^n = \sum_{k_1, k_2 \geq 0} c_n(k_1, k_2) \, x^{k_1} y^{k_2}.$$

Therefore

$$\log(1 + x + y + xy) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} (x + y + xy)^n$$

$$= \sum_{n \geq 1} \sum_{k_1, k_2 \geq 0} \frac{(-1)^{n+1}}{n} c_n(k_1, k_2) \, x^{k_1} y^{k_2} = \sum_{k_1, k_2 \geq 0} \left(\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} c_n(k_1, k_2) \right) x^{k_1} y^{k_2}.$$

Since

$$\log(1 + x + y + xy) = \log((1 + x)(1 + y)) = \log(1 + x) + \log(1 + y),$$

the terms with $k_1 \geq 1, k_2 \geq 1$ in the above series must vanish. This gives Formula 11.2.1. □

11.3. **The injectivity of $\nabla : \mathcal{E} \to R[\mathcal{S}_0]$.** We begin by associating with any virtual string α an element $\zeta(\alpha) \in \mathcal{E}$. Let S be the core circle of α. A *surgery* along an arrow $(a, b) \in \mathrm{arr}(\alpha)$ consists in picking two (positively oriented) arcs $aa^+, bb^+ \subset S$ and then quotienting the complement of their interiors $S - ((aa^+)^\circ \cup (bb^+)^\circ)$ by $a = b^+, b = a^+$. It is understood that the arcs aa^+, bb^+ are small enough not to contain endpoints of α besides a, b, respectively. Such a surgery transforms S into two disjoint oriented circles. We make each of them into a string by adding all the arrows of α with both endpoints on the arc a^+b (resp. on ba^+). (The arrows of α with one endpoint on ab and the other one on ba disappear under surgery.)

Let us call a set $F \subset \mathrm{arr}(\alpha)$ *special* if the arrows of α belonging to F are pairwise unlinked. Applying surgery inductively to all arrows of α belonging to a special set F, we transform α into $n = \#(F) + 1$ strings. Providing all the arrows of these strings with sign $+$, we obtain n arrow diagrams $D_1^F, ..., D_n^F$. Note that they have together at most $\#(\mathrm{arr}(\alpha)) - \#(F)$ arrows. We now define an oriented graph Γ_F. The vertices of Γ_F are the symbols $v_1, ..., v_n$. Two verices v_i, v_j are related by an oriented edge leading from v_i to v_j if there is an arrow $(a, b) \in F$ such that the arcs $aa^+, bb^+ \subset S$ involved in the surgery along this arrow lie on the core circles of D_i^F, D_j^F, respectively. It is easy to see that Γ_F is a tree. Set

$$\zeta(\alpha) = \sum_{F \subset \mathrm{arr}(\alpha)} \eta(\Gamma_F) \, z^{\#(F)} \prod_{i=1}^{\#(F)+1} [D_i^F] \in \mathcal{E}$$

where F runs over all special subsets of $\mathrm{arr}(\alpha)$. The summand corresponding to $F = \emptyset$ is the string α itself with sign $+$ on all arrows.

The key property of $\zeta(\alpha) \in \mathcal{E}$ is its invariance under the basic homotopy moves on α. This follows from [Tu2], Lemma 15.1.1 in the case where the moves are realized geometrically by homotopy of a curve realizing α on a surface. Since the homotopy moves can be always realized geometrically, the result follows. The mapping $\alpha \mapsto \zeta(\alpha)$ extends by multiplicativity to an algebra homomorphism $R[\mathcal{S}_0] \to \mathcal{E}$ denoted also ζ.

We can now prove the injectivity of ∇. For $r \geq 0$, denote by B_r the R-submodule of $\mathcal{E}$ additively generated by monomials $[D_1][D_2]\cdots[D_n]$ such that the total number of arrows in the arrow diagrams $D_1, D_2, ..., D_n$ is less than or equal to r. Clearly, $0 = B_0 \subset B_1 \subset ...$ and $\cup_r B_r = \mathcal{E}$. Pick $b = [D_1][D_2]\cdots[D_n] \in B_r$. Using the skein relation in $\mathcal{E}$ it is easy to see that $b (\mathrm{mod}\, B_{r-1}) \in B_r/B_{r-1}$ does not depend on the signs of the arrows of $D_1, ..., D_n$. This observation, Formula 10.3.1 and the definition of ζ imply that $(\zeta\nabla)(b) - b \in B_{r-1}$. Therefore $(\zeta\nabla - \mathrm{id})^r(b) = 0$. The inclusion $b \in \mathrm{Ker}\,\nabla$ would imply $b = 0$. Thus $B_r \cap \mathrm{Ker}\,\nabla = 0$. Since $\cup_r B_r = \mathcal{E}$, we obtain $\mathrm{Ker}\,\nabla = 0$.

11.4. **More on $\mathcal{E}$.** The comultiplication Δ defined in Section 11.1 makes $\mathcal{E}$ into a Hopf algebra over R. Its counit is the homomorphism $\varepsilon : \mathcal{E} \to R$ used in the definition of ∇. For an arrow diagram D, denote by $\tilde{D}$ the same diagram with opposite signs on all arrows. The transformation $[D] \mapsto -[\tilde{D}]$ preserves the skein relation and therefore induces an algebra automorphism of $\mathcal{E}$. This automorphism is an antipode for $\mathcal{E}$. This follows from the corresponding theorem for the skein algebras of curves on surfaces conjectured in [Tu2] and proven in [CR] and independently in [Pr]. In the construction of the Hopf algebra $\mathcal{E}$ instead of the ground ring $R = \mathbb{Q}[z]$ we can use $\mathbb{Z}[z]$. It is only to construct the homomorphisms ∇ and ζ that we need $\mathbb{Q}$.

Consider the Hopf algebra $S_z(\mathcal{A}_0)$ derived as in Section 9.6 from the spiral Lie coalgebra $\mathcal{A}_0$, the ring $R = \mathbb{Q}[z]$ and the element $h = z \in R$. Note that $S_h(\mathcal{A}_0) = R[\mathcal{S}_0]$ as algebras.

Theorem 11.4.1. *The homomorphism $\nabla : \mathcal{E} \to R[\mathcal{S}_0] = S_z(\mathcal{A}_0)$ is an isomorphism of Hopf algebras.*

The proof of this theorem follows the lines of [Tu2], Section 12 and Lemma 13.5; we omit the details.

12. Open strings

12.1. **Definitions.** Replacing the circle in the definition of a virtual string by an oriented one-dimensional manifold X we obtain a *virtual string with core manifold* X. The definition of homotopy extends to strings with core manifold X word for word. Of special interest are strings with core manifold homeomorphic to $[0, 1]$; we call them *open strings*. In this context it is natural to call virtual strings with core manifold homeomorphic to S^1 *closed* strings.

Open strings underlie (generic) paths on surfaces connecting distinct points on the boundary. Gluing the endpoints of the core interval, we can transform any open string μ into a closed string μ^{cl}, its *closure*. Similarly to paths, open strings can be multiplied via the gluing of their core intervals along one endpoint. Repeating word for word the definitions of Section 3.6 we obtain for all $r \geq 1$ the notion of an r-th covering of an open string (this is again an open string).

12.2. **Polynomials of open strings.** For an open string μ with core manifold $[0, 1]$, we can define two polynomials $u^+(\mu)$ and $u^-(\mu)$. Observe that the set $\mathrm{arr}(\mu)$ of arrows of μ is a disjoint union $\mathrm{arr}^+(\mu) \cup \mathrm{arr}^-(\mu)$ where $\mathrm{arr}^+(\mu)$ (resp. $\mathrm{arr}^-(\mu)$) is the set of arrows $(a, b) \in \mathrm{arr}(\mu)$ with $a, b \in [0, 1]$ such that $a < b$ (resp. $b < a$). For $e \in \mathrm{arr}(\mu)$, set $n(e) = n(e^{cl}) \in \mathbb{Z}$ where e^{cl} is the corresponding arrow of μ^{cl}. For $k \geq 1$, set

$$u_k^\pm(\mu) = \#\{e \in \mathrm{arr}^\pm(\mu) \,|\, n(e) = k\} - \#\{e \in \mathrm{arr}^\mp(\mu) \,|\, n(e) = -k\} \in \mathbb{Z}.$$

This number and the polynomials $u^\pm(\mu) = \sum_{k\geq 1} u_k^\pm(\mu)\, t^k$ are homotopy invariants of μ. Clearly, $u(\mu^{cl}) = u^+(\mu) + u^-(\mu)$ and $u^\pm(\mu\nu) = u^\pm(\mu) + u^\pm(\nu)$ for any open strings μ, ν. Using $u^\pm$, it is easy to give examples of non-homotopic open strings with homotopic closures. Using the coverings as in Section 3.6, we can define for open strings "higher versions" of $u^\pm$ parametrized by finite sequences of positive integers.

12.3. **Cobordism of open strings.** An open string μ is *slice* if its closure μ^{cl} is a slice (closed) string. Theorem 5.1.2 implies that the sliceness is a homotopy property of open strings.

An open string μ is *ribbon* if its core interval admits an orientation reversing involution transforming arrows of μ into arrows of μ with opposite orientation. The closure of a ribbon open string is a ribbon (closed) string. Therefore ribbon open strings are slice.

We associate with every open string μ an open string μ' obtained from μ by reversing orientation on the core interval and on all arrows. Clearly $(\mu')' = \mu$ and $(\mu\nu)' = \nu'\mu'$ for any open strings μ, ν.

We say that open strings μ, ν are *cobordant* and write $\mu \sim_c \nu$ if $\mu\nu'$ is slice. An open string is cobordant to a trivial open string (with no arrows) if and only if it is slice.

Lemma 12.3.1. *(i) Cobordism is an equivalence relation on the set of open strings.*

(ii) Homotopic open strings are cobordant.

(iii) If two open strings are cobordant, then their closures are cobordant.

(iii) If two open strings are cobordant, then their r-th coverings are cobordant for all $r \geq 1$.

Proof. For any open string μ, the product $\mu\mu'$ is ribbon and therefore slice. Thus $\mu \sim_c \mu$. If $\mu \sim_c \nu$, then $(\mu\nu')^{cl}$ is slice. The closed string $(\nu\mu')^{cl}$ is obtained from $(\mu\nu')^{cl} = ((\nu\mu')')^{cl}$ by the involution $\alpha \mapsto \overline{\alpha}^-$. Hence $(\nu\mu')^{cl}$ is slice and $\nu \sim_c \mu$.

To proceed we need the following property: if μ, ν, δ are open strings whose product $\mu\nu\delta$ is slice and if ν is slice, then so is $\mu\delta$. Indeed, observe that $(\mu\nu\delta)^{cl}$ is homeomorphic to $(\nu\delta\mu)^{cl}$. Since $(\nu\delta\mu)^{cl}$ is a product of ν^{cl} and $(\delta\mu)^{cl}$, the cancellation property mentioned in Section 5.2 implies that $(\delta\mu)^{cl}$ is slice. Since $(\delta\mu)^{cl}$ is homeomorphic to $(\mu\delta)^{cl}$, the latter string is slice. Hence $\mu\delta$ is slice.

We can now prove the transitivity of cobordism. If $\mu \sim_c \nu$, $\nu \sim_c \delta$, then $\mu\nu'$ and $\nu\delta'$ are slice. Since products of slice closed strings are slice, the products of slice open strings are slice. Thus, $\mu\nu'\nu\delta'$ is slice. Since $\nu'\nu$ is slice, so is $\mu\delta'$. Therefore $\mu \sim_c \delta$.

If μ is homotopic to ν, then $(\mu\nu')^{cl}$ is homotopic to $(\nu\nu')^{cl}$. Therefore $(\mu\nu')^{cl}$ is slice so that $\mu \sim_c \nu$. This proves (ii). We leave (iii) and (iv) as an exercise for the reader. □

Multiplication of open strings induces a multiplication in the set of cobordism classes of open strings that makes this set into a group denoted $\mathcal{O}$. Using the formulas $u^\pm(\mu') = -u^\pm(\mu)$ and Theorem 5.1.4 we obtain that $u(\mu^{cl}) = u^+(\mu) + u^-(\mu) \in \mathbb{Z}[t]$ is an additive cobordism invariant of open strings.

12.4. **Graded based matrices.** A *graded (skew-symmetric) based matrix* over an abelian group H is a based (skew-symmetric) matrix (G, s, b) over H endowed with a spltting of $G - \{s\}$ as a union of two disjoint subsets G^+ and G^-. The *underlying based matrix* of a graded based matrix (G, s, b) is obtained by forgetting the splitting $G - \{s\} = G^+ \cup G^-$. The *negation* $-T$ of $T = (G, s, b)$ is the triple $(G, s, -b)$ with the same splitting $G - \{s\} = G^+ \cup G^-$.

We define annihilating elements, core elements, and complementary elements of a graded based matrix $T = (G, s, b)$ as in Section 6.1 with the following additional requirements: annihilating elements must lie in G^+, core elements must lie in G^- and for any pair of complementary elements, one of them lies in G^+ and the second one in G^-. All other dfefinitions and results of Section 6.1 extend to this setting with obvious changes. In particular, we have a notion of homology for graded based matrices over H.

If $H \subset \mathbb{R}$, then the two 1-variable polynomials

$$u^\pm(T)\,(t) = \sum_{g \in G^\pm, b(g,s) > 0} t^{b(g,s)} - \sum_{g \in G^\mp, b(g,s) < 0} t^{-b(g,s)}$$

are homology invariants of a graded based matrix $T = (G, s, b)$.

For an open string μ, the based matrix of its closure μ^{cl} is graded via the splitting $\operatorname{arr}(\mu^{cl}) = \operatorname{arr}(\mu) = \operatorname{arr}^+(\mu) \cup \operatorname{arr}^-(\mu)$. This defines a graded based matrix $T(\mu)$ over $\mathbb{Z}$. The homology class of $T(\mu)$ is an invariant of the homotopy class of μ. Clearly, $u^\pm(\mu) = u^\pm(T(\mu))$. Note also that $T(\mu') = -T(\mu)$.

12.5. **Addition of graded based matrices.** We define addition for graded based matrices which mimics the product of strings. Let $T_i = (G_i, s_i, b_i)$ be a graded based matrix over an abelian group H where $i = 1, 2$. We define the sum $T_1 \oplus T_2 = (G, s, b)$ as follows. Set $G^\pm = G_1^\pm \amalg G_2^\pm$ and $G = \{s\} \amalg G^+ \amalg G^-$. For $g \in G - \{s\} = G^+ \amalg G^-$, set $\varepsilon_g = 1$ if $g \in G^+$ and $\varepsilon_g = 0$ if $g \in G^-$. The skew-symmetric mapping $b : G \times G \to H$ is defined as follows: for $g \in G_i - \{s_i\}$ with $i \in \{1, 2\}$, set $b(s, g) = b_i(s_i, g), b(g, s) = b_i(g, s_i)$; for any $g \in G_i - \{s_i\}, h \in G_j - \{s_j\}$ with $i, j \in \{1, 2\}$, set

$$b(g, h) = \begin{cases} b_i(g, h), \text{ if } i = j, \\ \varepsilon_g b_j(s_j, h) - \varepsilon_h b_i(s_i, g), \text{ if } i \neq j. \end{cases}$$

The direct sum of graded based matrices is commutative and associative (up to isomorphism).

Let R be a domain. A graded based matrix over R is *hyperbolic* if its underlying based matrix is hyperbolic.

Lemma 12.5.1. *For any graded based matrix over R, its direct sum with its negation is hyperbolic. The direct sum of two hyperbolic graded based matrices over R is hyperbolic.*

Proof. Let $T_1 = (G_1, s_1, b_1)$ be a graded based matrix over R. Let $T_2 = (G_2, s_2, b_2)$ be a copy of T_1 where $G_2 = \{g' \mid g \in G_1\}$, $s_2 = (s_1)'$, and b_2 is defined by $b_2(g', h') = b(g, h)$ for $g, h \in G_1$. We verify that the direct sum $T = T_1 \oplus (-T_2) = (G, s, b)$ is hyperbolic. Consider the subsets $\{s\}$ and $\{g, g'\}_{g \in G_1 - \{s_1\}}$ of G. These subsets form a simple filling of G. The matrix of this filling is zero. Indeed, for $g \in G_1 - \{s_1\}$,

$$b(s, \{g, g'\}) = b(s, g) + b(s, g') = b_1(s_1, g) + (-b_2)(s_2, g') = b_1(s_1, g) - b_2((s_1)', g') = 0.$$

For $g, h \in G_1 - \{s_1\}$,

$$\begin{aligned} b(\{g, g'\}, \{h, h'\}) &= b(g, h) + b(g, h') + b(g', h) + b(g', h') \\ &= b_1(g, h) + \varepsilon_g(-b_2)(s_2, h') - \varepsilon_{h'} b_1(s_1, g) + \varepsilon_{g'} b_1(s_1, h) - \varepsilon_h(-b_2)(s_2, g') + (-b_2)(g', h') \\ &= b_1(g, h) - \varepsilon_g b_1(s_1, h) - \varepsilon_h b_1(s_1, g) + \varepsilon_g b_1(s_1, h) + \varepsilon_h b_1(s_1, g) - b_1(g, h) = 0. \end{aligned}$$

The second claim of the lemma is an exercise on the definitions; we leave it to the reader. □

Quotienting the monoid of graded based matrices over R by hyperbolic matrices, we obtain an abelian group $\mathcal{G}(R)$. We call it the *group of cobordisms* of graded based matrices over R.

Assigning to an open string its graded based matrix we obtain an additive homomorphism $\mathcal{O} \to \mathcal{G}(\mathbb{Z})$. The group $\mathcal{G}(\mathbb{Z})$ is non-trivial. This is clear from the existence of non-trivial additive homomorphisms $u^{\pm} : \mathcal{G}(\mathbb{Z}) \to \mathbb{Z}[t]$.

The notion of a graded based matrix over R and the addition of such matrices may seem artificial from the algebraic viewpoint. Possibly, a more satisfactory (although equivalent) language would describe a graded based matrix over R as a free R-module V of finite rank endowed with a vector in the dual module $V^* = \mathrm{Hom}_R(V, R)$, with a distinguished basis partitioned into two disjoint subsets, and with a R-valued skew-symmetric bilinear form $V \times V \to R$. To pass from the definition above to this one, we associate with (G, s, b) the free R-module V with basis $G - \{s\}$, the element of V^* sending any $g \in G - \{s\}$ to $b(s, g)$, the partition $G - \{s\} = G^+ \cup G^-$, and the skew-symmetric bilinear form $V \times V \to R$ induced by b.

12.6. **The algebra of open strings.** Let R be a commutative ring with unit and $\otimes = \otimes_R$. A (left) *module* over a Lie algebra $(L, [\,,] : L^{\otimes 2} \to L)$ over R is an R-module M endowed with an R-linear homomorphism $\rho : L \otimes M \to M$ such that

$$\rho([\,,] \otimes \mathrm{id}_M) = \rho(\mathrm{id}_L \otimes \rho)(\mathrm{id}_{L \otimes L \otimes M} - \mathrm{Perm}_L \otimes \mathrm{id}_M) : L \otimes L \otimes M \to M \tag{12.6.1}$$

where Perm_L is the permutation $x \otimes y \mapsto y \otimes x$ in $L^{\otimes 2}$. (Formula 12.6.1 is equivalent to the usual identity $[x, y]m = x(ym) - y(xm)$ for $x, y \in L, m \in M$.) Dually, a *comodule* over a Lie coalgebra $(A, \nu : A \to A^{\otimes 2})$ over R is an R-module M endowed with an R-linear homomorphism $\rho : M \to A \otimes M$ such that

$$(\nu \otimes \mathrm{id}_M)\rho = (\mathrm{id}_{A \otimes A \otimes M} - \mathrm{Perm}_A \otimes \mathrm{id}_M)(\mathrm{id}_A \otimes \rho)\rho : M \to A \otimes A \otimes M. \tag{12.6.2}$$

Such M is automatically a module over the dual Lie algebra $A^* = \mathrm{Hom}_R(M, R)$: an element $a \in A^*$ acts on M by the endomorphism $\varphi_a : M \to M$ sending $m \in M$ to $-(a \otimes \mathrm{id}_M)\rho(m) \in R \otimes M = M$.

A comodule (M, ρ) over a Lie coalgebra A is *spriral* if $M = \cup_{n \geq 1} \mathrm{Ker}\, \rho^{(n)}$ where

$$\rho^{(n)} = (\mathrm{id}_A^{\otimes (n-1)} \otimes \rho) \circ \cdots \circ (\mathrm{id}_A \otimes \rho) \circ \rho : M \to A^{\otimes n} \otimes M.$$

If $R \supset \mathbb{Q}$ and A, M are spiral, then the action of A^* on M integrates into a group action of the group $\mathrm{Exp}\, A^*$ on M: an element $a \in \mathrm{Exp}\, A^* = A^*$ acts on $m \in M$ by

$$am = e^{\varphi_a}(m) = m + \sum_{k \geq 1} (\varphi_a)^k(m)/k!.$$

Note that for $m \in \mathrm{Ker}\, \rho^{(n)}$ the sum on the right-hand side has at most n non-zero terms.

Let $\mathcal{M} = \mathcal{M}(R)$ be the free R-module freely generated by the set of homotopy classes of open virtual strings. We provide $\mathcal{M}$ with the structure of a comodule over the Lie coalgebra of closed strings $\mathcal{A}_0$. Let $\langle \mu \rangle$ be the generator of $\mathcal{M}$ represented by an open string μ. For an arrow $e \in \mathrm{arr}(\mu)$, a surgery along e defined as in Section 11.3 transforms μ into a disjoint union of a closed string α_e and an open string β_e. Set

$$\rho(\langle \mu \rangle) = \sum_{e \in \mathrm{arr}_+(\mu)} \langle \alpha_e \rangle \otimes \langle \beta_e \rangle - \sum_{e \in \mathrm{arr}_-(\mu)} \langle \alpha_e \rangle \otimes \langle \beta_e \rangle \in \mathcal{A}_0 \otimes \mathcal{M}.$$

A direct computation shows that this gives a well-defined R-linear homomorphism $\rho : \mathcal{M} \to \mathcal{A}_0 \otimes \mathcal{M}$ satisfying Formula 12.6.2. Thus $\mathcal{M}$ is a comodule over $\mathcal{A}_0$. Combining ρ with the inclusion $\mathcal{A}_0 \subset \mathcal{A}$ we obtain that $\mathcal{M}$ is a comodule over $\mathcal{A}$ as well. It is easy to see that $\mathcal{M}$ is spiral. If $R \supset \mathbb{Q}$, then the construction above gives a group action of $\operatorname{Exp} \mathcal{A}^*$ on $\mathcal{M}$.

12.7. **Exercises.** 1. Multiplication of open strings makes $\mathcal{M}$ into an associative algebra with unit. Check that the group $\operatorname{Exp} \mathcal{A}^*$ acts on $\mathcal{M}$ by algebra automorphisms.

2. Let $cl : \mathcal{M} \to \mathcal{A}$ be the R-linear homomorphism induced by closing open strings. Check that for any open string μ, we have $\nu(\langle cl(\mu)\rangle) = (\mathrm{id}_{\mathcal{A}\otimes\mathcal{A}} - \mathrm{Perm}_{\mathcal{A}})(\mathrm{id}_{\mathcal{A}} \otimes cl)\rho(\langle \mu \rangle)$.

13. Questions

1. Which primitive based matrices $T_\bullet$ can be realized as $T_\bullet(\alpha)$ for a string α ? A necessary condition pointed out in Section 3.2 says that $(u(T_\bullet))'(1) = 0$. Note that for the based matrix $T(\alpha) = (G, s, b)$, we have $|b(e, f)| \leq \#(G) - 2$ for all $e, f \in G$. This however yields no conditions on the primitive based matrices of strings, since such a matrix $T_\bullet = (G_\bullet, s_\bullet, b_\bullet)$ may arise from a string of a rank $\gg \#(G_\bullet)$.

2. Can one detect non-slice strings with hyperbolic based matrices using the secondary obstructions of Section 8.4 ?

3. Is it true that slice strings are stably ribbon, i.e., that for any slice string α there is a ribbon string β such that a product of α and β is homotopic to a ribbon string ? Is it true for open strings ? A positive answer to the second question would imply a positive answer to the first question.

4. Classify all strings of small rank (say, ≤ 6) up to homotopy and/or up to cobordism.

5. Is it true that every string is homotopic to a string of type α_σ for some permutation σ ? If not, is it true up to cobordism ?

6. Is multiplication of open strings commutative up to homotopy ? If not, is it commutative up to cobordism ?

7. Compute the group $\mathcal{O}$ of cobordism classes of open strings.

8. Compute the group $\mathcal{G}(\mathbb{Z})$ of cobordism classes of graded based matrices over $\mathbb{Z}$.

9. Is there an invariant of virtual knots combining the skein invariant ∇ with the Kontsevich universal finite type invariant of knots ? This might lead to mixed arrow-chord diagrams.

10. Study invariants of virtual strings that change in a controlled way (say by constants) under the moves $(\mathrm{a})_s$, $(\mathrm{b})_s$, $(\mathrm{c})_s$, cf. the theory of Arnold's invariants of plane curves.

11. Generalize the invariants of virtual knots introduced in this paper to virtual links.

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IRMA, Université Louis Pasteur - C.N.R.S.,
7 rue René Descartes
F-67084 Strasbourg
France